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Motion of a graph by convexified energy

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1. Introduction

This paper is concerned with evolution equations of hypersurfaces Γ_t in $\boldsymbol{R}^n$. We consider

$$V=\frac{-1}{\beta(\vec{n})}\left(\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial\gamma}{\partial p_i}(\vec{n})\right)+c\right) \quad \text{on } \Gamma_t. \tag{1.1}$$

Here $\vec{n}$ represents the unit normal vector (field) of Γ_t and V represents the normal velocity of Γ_t. The function $\gamma=\gamma(p_1, \cdots, p_n)$ is assumed to be positively homogeneous of degree one and its restriction on the unit sphere S^{n-1} is often called the interface energy density. The function $\beta: S^{n-1}\rightarrow \boldsymbol{R}$ is assumed to be positive and continuous; c is a constant. The sign in front of $1/\beta$ is taken so that the equation (1.1) becomes the mean curvature fiow equation if $\gamma(p)=|p|$, $\beta\equiv 1$ and $c=0$. The equation (1.1) is considered as a mathematical model for the dynamics of surfaces of a melting solid when the effect outside the surface is negligible. We refer to a paper [AG 1] of Angenent and Gurtin for its derivation from the second law in the thermodynamics and the force balances.

Usually, γ is assumed to be convex and C^2 outside the origin. However, in physics there is also the possibility that γ is not convex as studied in [AG 1,2]. If γ is not convex, the equation (1.1) is not well-posed even locally because it is backward parabolic in some direction of $\vec{n}$. To track the evolution of the hypersurface it seems to be natural to consider the convexification $\tilde{\gamma}$ of γ when γ is not convex. We are interested in the evolution of the hypersurface by (1.1) where γ is replaced by $\tilde{\gamma}$.

In this paper we consider the evolution when the hypersurface Γ_t is a curve represented by the graph of a function on $\boldsymbol{R}$. Even in this simple case there arise several problems. First, solution Γ_t may develop singularities in a finite time because $\tilde{\gamma}$ may not be strictly convex. Second, $\tilde{\gamma}$ may not be C^2 even if γ is smooth, so the interpretation of (1.1) is not clear. Instead of considering general convexified $\tilde{\gamma}$ we restrict ourselves to handle typical one by assuming that $\tilde{\gamma}$ is not C^2 at most in finitely many directions and the gradient of $\tilde{\gamma}$ is locally Lipschitz outside zero.