Hokkaido Mathematical Journal Vol. 24 (1995) p. 567–573

Exotic circles of $PL_{+}(S^{1})$

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(Received January 6, 1995; Revised March 3, 1995)

Abstract. Let G be a subgroup of $\mathrm{Homeo}_{+}(S^{1})$. An exotic circle of G is a subgroup of G which is topologically conjugate to $SO(2)$ but not conjugate to $SO(2)$ in G. This shows us the subgroup G is far from being a Lie group. In this paper, we prove that $PL_{+}(S^{1})$ has exotic circles.

Key words: exotic circle, $PL_{+}(S^{1})$, topologically conjugate, PL conjugate, bendeng point.

Introduction

Let G be a Lie group and M an oriented manifold of class $C^{k}(1\leq k\leq\infty)$. Let $\mathrm{Diff}_{+}^{k}(M)$ denote the group of all C^{k} diffeomorphisms of M. A topological action is a continuous map $\varphi: G\times M\rightarrow M$ such that
1) $\varphi_{e}(x)=x$,
2) $\varphi_{gh}(x)=\varphi_{g}(\varphi_{h}(x))$.
where e is the unit of G and $\varphi_{g}(x)=\varphi(g, x)$. D. Montgomery and L. Zippin proved the following theorem ([4]).

Theorem 0.1 *Let φ be a topological action. If every φ_{g} belongs to $\mathrm{Diff}_{+}^{k}(M)$ then φ is a map of class C^{k}.*

In the case where $G=M=S^{1}$, this theorem implies the following corollary.

Corollary 0.2 *If every $h\circ R_{x}\circ h^{-1}$ is contained in $\mathrm{Diff}_{+}^{k}(S^{1})$, then h belongs to $\mathrm{Diff}_{+}^{k}(S^{1})$. Here, $R_{x}: S^{1}\rightarrow S^{1}$ is the rotation of S^{1}, i.e., $R_{x}(y)=x+y$.*

Indeed, for $\varphi(x, y)=h\circ R_{x}\circ h^{-1}(y)$. $\varphi: S^{1}\times S^{1}\rightarrow S^{1}$ is a topological action with $\varphi_{x}\in \mathrm{Diff}_{+}^{k}(S^{1})$. Then φ is of class C^{k} by Theorem 0.1. Fix a point y_{0} and define the C^{k} diffeomorphism ϕ of S^{1} by $\phi(x)=\varphi(x, y_{0})$. Then we can see easily $\phi^{-1}\circ\varphi_{x}\circ\phi=R_{x}$. So $\phi^{-1}\circ h=R_{z}$ for some $z\in S^{1}$. This implies h belongs to $\mathrm{Diff}_{+}^{k}(S^{1})$.

Let $SO(2)=\{R_{x}|x\in S^{1}\}$ be the group of all rotations of S^{1}. Corollary

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1991 Mathematics Subject Classification : 58E40, 58F03.