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Grötzsch ring and quasiconformal distortion functions

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Abstract. The authors obtain improved estimates for the modulus of the Grötzsch ring, derive sharp bounds for the Schwarz distortion function in the plane, and indicate some extensions to higher dimensions.

Key words: conformal capacity, distortion, inequalities, Grötzsch ring, modulus, quasiconformal.

1. Introduction and Notation

Let $R_{G,n}(s)$ denote the Grötzsch ring in $\mathbb{R}^n$, $n \geq 2$, which is bounded by the unit sphere S^{n-1} and the ray $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > s,\ x_j = 0,\ 2 \leq j \leq n\}$, $s > 1$. The conformal capacity of $R_{G,n}(s)$ is denoted by

$$\gamma_n(s) = \operatorname{cap} R_{G,n}(s), \tag{1.1}$$

and the modulus $M_n(r)$ of $R_{G,n}(1/r)$, $0 < r < 1$, is defined by

$$M_n(r) = [\omega_{n-1}/\gamma_n(1/r)]^{1/(n-1)}, \tag{1.2}$$

where ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$. These functions are important in the study of distortion properties of quasiconformal mappings [G, I, Vu1-2, AVV1-4, P1].

The function $M_2(r)$ is usually denoted by $\mu(r)$, and has the explicit expression [LV, p. 60]

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}, \tag{1.3}$$

where

$$\mathcal{K}(r) = \int_0^{\frac{\pi}{2}} (1 - r^2 \sin^2 t)^{-\frac{1}{2}}\, dt, \quad \mathcal{K}'(r) = \mathcal{K}(r'),$$

$r' = (1 - r^2)^{\frac{1}{2}}$, $0 < r < 1$, are complete elliptic integrals of the first kind [BF, Bo, BB]. We also need the complete elliptic integrals of the second

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