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Anisotropic motion by mean curvature in the context of Finsler geometry

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Abstract. We study the anisotropic motion of a hypersurface in the context of the geometry of Finsler spaces. This amounts in considering the evolution in relative geometry, where all quantities are referred to the given Finsler metric ϕ representing the anisotropy, which we allow to be a function of space. Assuming that ϕ is strictly convex and smooth, we prove that the natural evolution law is of the form "velocity $= H_{\phi}$", where H_{ϕ} is the relative mean curvature vector of the hypersurface. We derive this evolution law using different approches, such as the variational method of Almgren-Taylor-Wang, the Hamilton-Jacobi equation, and the approximation by means of a reaction-diffusion equation.

Key words: Finsler spaces, mean curvature flow, fronts propagation, surface area.

1. Introduction

The concepts of surface energy, particularly that of anisotropic surface energy and of related quantities such as the anisotropic mean curvature, are becoming increasingly important in different contexts, as in the field of phase changes and phase separation in multiphase materials [1], [29]. The role played by anisotropy becomes crucial in the crystalline case [2], [3], [12], [42], [43], [44], where the principal curvatures in the sense of differential geometry cannot in general be defined pointwise everywhere [40]. However the study of anisotropic evolution problems in the smooth case is a first step for a better understanding of the role of anisotropy in the general case in which no differentiability properties are assumed.

Anisotropic surface energy falls quite naturally within the geometry of Finsler spaces [6], [8], [32], and many of the tools of convex geometry [39] prove useful for related variational problems [9]. In particular, the idea is to endow the space $\mathbf{R}^N$ with the distance obtained by integrating the Finsler

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