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Extremal rings and quasiconformal mappings

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Abstract. New functional inequalities are obtained for the capacities of Grötzsch and Teichmüller rings, and for complete elliptic integrals, thus solving two conjectures. These results are applied to refine earlier estimates in quasiconformal Schwarz lemma and Mori's theorem.

Key words: conformal capacity, distortion, modulus, quasiconformal, ring.

1. Introduction

For $n \geq 2$, $s > 1$ and $t > 0$, let $R_{G,n}(s)$ denote the Grötzsch ring in $\mathbf{R}^n$, whose complementary components are the closed unit ball $\overline{B}^n$ and the ray $[s, \infty)$ along the x_1-axis, and let $R_{T,n}(t)$ denote the Teichmüller ring in $\mathbf{R}^n$, whose complementary components are the segment $[-1, 0]$ and the ray $[t, \infty)$ along the x_1-axis. The conformal capacities of $R_{G,n}(s)$ and $R_{T,n}(t)$ are denoted by

$$\gamma_n(s) = \text{cap } R_{G,n}(s), \quad \tau_n(t) = \text{cap } R_{T,n}(t), \tag{1.1}$$

respectively. The modulus $M_n(r)$ of the Grötzsch ring $R_{G,n}(1/r)$, $0 < r < 1$, is defined by

$$M_n(r) = [\omega_{n-1}/\gamma_n(1/r)]^{1/(n-1)}, \tag{1.2}$$

where ω_{n-1} is the $(n-1)$-dimensional measure of the unit sphere S^{n-1} in $\mathbf{R}^n$ [G], [Vä], [Vu1]. The capacities in (1.1) are related [G, §18] by

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1), \quad s > 1. \tag{1.3}$$

For $K > 0$, define the increasing homeomorphism $\varphi_{K,n}(r)$ from $[0, 1]$ onto $[0, 1]$ by

$$\varphi_{K,n}(r) = 1/\gamma_n^{-1}(K\gamma_n(1/r)) = M_n^{-1}(\alpha M_n(r)) \tag{1.4}$$

for $r \in (0, 1)$, $\varphi_{K,n}(0) = \varphi_{K,n}(1) - 1 = 0$, where $\alpha = K^{1/(1-n)}$. These functions arise in the study of quasiconformal mappings in $\mathbf{R}^n$ [AVV1]–

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