Hokkaido Mathematical Journal Vol. 26 (1997) p. 125–139

Symmetry of isometric embeddings of Riemannian manifolds and local scalar invariants*

Chong-Kyu HAN and Jae-Nyun YOO

(Received April 26, 1995; Revised January 29, 1996)

Abstract. We study the infinitesimal symmetry of the isometric embeddings of a Riemannian manifold M^n into $\mathbb{R}^{n+d}$, $n \geq 2$, $d \geq 1$. Then we define a notion of scalar invariant for submanifolds in $\mathbb{R}^{n+d}$ in terms of this symmetry. As an example, we show by calculation that the Gaussian curvature of a surface is an invariant.

Key words: isometric embedding, infinitesimal symmetry, scalar invariant, Killing field.

Introduction

Let M be a smooth (C^∞) manifold of dimension n, $n \geq 2$, with Riemannian metric g. A mapping $u = (u^1, \dots, u^{n+d}) : M \to \mathbb{R}^{n+d}$, $d \geq 1$, is a local isometric embedding if u satisfies

$$\langle du, du \rangle = g.$$

In terms of local coordinates $x = (x^1, \dots, x^n)$ of M, the above equation is written as

$$\sum_{\alpha=1}^{n+d} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} = g_{ij}, \quad \text{for each} \quad i, j = 1, \dots, n, \tag{2.5}$$

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

A (local) scalar invariant of M is a real valued function defined on an open subset of M which is invariant under local isometries. The scalar curvature is the simplest scalar invariant. If $\mathrm{vol}_M(p, r)$ is the volume of the geodesic ball of radius r centered at a point $p \in M$, then for sufficiently small $r \geq 0$

$$\frac{\mathrm{vol}_M(p, r)}{\mathrm{vol}_{\mathbb{R}^n}(0, r)} = 1 - c\kappa_2(p) r^2 + \sum_{n \geq 4} \kappa_n(p) r^n,$$

1991 Mathematics Subject Classification : 53A55, 58A20, 58G35.

*The first author was partially supported by Korea Research Foundation's program at POSTECH and both authors were partially supported by GARC-KOSEF.