Hokkaido Mathematical Journal Vol. 26 (1997) p. 1–61

Geodesic and metric completeness in infinite dimensions

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(Received May 30, 1995)

Abstract. Some infinite-dimensional Riemannian manifolds are constructed in which the induced metric is incomplete, but all geodesics may be indefinitely extended in both directions and any two points may be joined by a minimizing geodesic. The construction relies on some estimates for truncated Hilbert transforms.

Key words: Riemannian, geodesic.

1. Introduction

Let M be a connected C^{∞} Riemannian manifold without boundary, modelled on a Hilbert space H; the Riemannian structure g on M prescribes on each tangent space to M an inner product which defines the original topology on the tangent space. A metric d on M is determined by g, via path-lengths, in the usual way, and induces the original topology on M. M is described as "complete", or "metrically complete", if (M, d) is a complete metric space.

A "geodesic" in M is a solution of the geodesic equation, defined on an open interval in $\mathbb{R}$. A geodesic is *right-complete* if its domain is unbounded on the right; a complete geodesic has domain $(-\infty, +\infty)$, so that it is either constant or of infinite length in both directions. If every maximal geodesic through the point $x \in M$ is right-complete, say that M is "geodesically complete at x"; it is "geodesically complete" if every maximal geodesic is complete. M is geodesically complete if it is metrically complete (a disguised version of this easy fact is Lemma 2.6 below).

The standard proof (de Rham [17]; see [14], pp. 172–176, or [12], pp. 56–58, or [13], pp. 126–127, etc.) of the Hopf-Rinow theorem shows also that, if M is finite-dimensional and geodesically complete at x for some $x \in M$, then it is metrically complete. The proof evidently uses local compactness, and the result is indeed false in infinite dimensions. If M is complete but infinite-dimensional, there may be a pair of points $a, b \in M$ which cannot be joined by a geodesic (see [2]); then $M_1 = M \backslash \{b\}$ is geodesically complete

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1991 Mathematics Subject Classification : 58B20.