Hokkaido Mathematical Journal Vol. 27 (1998) p. 605–623

Irrational foliations of $S^3 \times S^3$

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(Received June 4, 1997)

Abstract. The Godbillon-Vey class is a characteristic cohomology class of dimension 3 for foliations of codimension 1 whose transition functions are transversely Lipschitz and with derivatives of bounded variations. We show that for a foliation $\mathcal{F}$ of $S^3 \times S^3$ of codimension 1, the ratio a/b of the Godbillon-Vey class $GV(\mathcal{F}) = (a, b) \in \mathbf{R} \oplus \mathbf{R} \cong H^3(S^3 \times S^3; \mathbf{R})$ takes any real value. It has been known that this ratio is invariant under the deformation of smooth foliations.

Key words: codimension 1 foliations, classifying spaces, Godbillon-Vey class, rationality.

Introduction

Let $\mathcal{F}$ be a codimension-one foliation of of $S^3 \times S^3$. For a codimension-one foliation, the Godbillon-Vey class is defined as a 3-dimensional cohomology class ([6]). Hence in this case, $GV(\mathcal{F}) \in H^3(S^3 \times S^3; \mathbf{R}) \cong \mathbf{R} \oplus \mathbf{R}$. We call $\mathcal{F}$ rational if $GV(\mathcal{F}) = (a, b) \in H^3(S^3 \times S^3; \mathbf{R})$ satisfies $a/b \in \mathbf{Q} \cup \{\infty\}$, and call $\mathcal{F}$ irrational if $a/b \in \mathbf{R} - \mathbf{Q}$. Gel'fand-Feigin-Fuks ([2]) noticed that this ratio a/b is invariant under a deformation of codimension-one foliations. Hence rationality or irrationality of foliations of $S^3 \times S^3$ is invariant under deformation.

This definition of rationality and irrationality imitates the one for the linear foliations of the 2-dimensional torus T^2. (See [12], [13] for the interesting progress in piecewise linear foliations on T^2.) A rational linear foliation of T^2 is defined by a submersion to the circle S^1. In a similar way, we can construct examples of rational foliations of $S^3 \times S^3$ by defining a Haefliger structure on $S^3 \times S^3$ as a pull-back by an appropriate map to S^3 and using the theorem of existence of foliations ([16]). An irrational linear foliation of T^2 is easy to construct. But it has not been known whether there exist irrational foliations of $S^3 \times S^3$. The question of the existence of irrational foliations of $S^3 \times S^3$ was raised in Gel'fand-Feigin-Fuks [2] and

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1991 Mathematics Subject Classification : Primary 57R32; Secondary 57R30, 57R50, 58H10.

The author was supported in part by Grant-in-Aid for Scientific research 07454013 and 09440028, Ministry of Education, Science, Sports and Culture, Japan.