Hokkaido Mathematical Journal Vol. 31 (2002) p. 349–361

A characterization of some finite simple groups by orders of their solvable subgroups

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(Received September 20, 2000)

Abstract. A brand new way to characterize a finite simple group is discovered. Some infinite sequences of non abelian simple groups have turned out to be determined just by each set of orders of solvable subgroups. The set of orders of solvable subgroups of the alternating group A_5 of degree 5 is $\{1,2,3,4,5,6,10,12\}$. But it is not very easy to see that no other non abelian simple group has such a set as above. In this paper we will study the problem above for any finite simple group.

Key words: Sylow subgroup, solvable subgroup, prime graphs, simple groups.

1. Introduction

Five kinds of infinite sequences of finite simple groups $A_1(q)$, ${}^2B_2(q)$, ${}^3D_4(q)$, $G_2(q)$ ${}^2G_2(q)$ are shown to be able to be determined just by each set of orders of solvable subgroups as described in the main theorem as follows.

Theorem 1 *Let S be one of the following non abelian finite simple groups*

$$A_1(q),\ Sz(q),\ {}^3D_4(q),\ G_2(q),\ {}^2G_2(q).$$

For a finite group G, if the set $\mathrm{ord}(\mathcal{S}_{sol}(G))$ *of orders of solvable subgroups of G coincides with* $\mathrm{ord}(\mathcal{S}_{sol}(S))$*, then G is isomorphic to S.*

We can regard this theorem as a kind of characterization of non abelian finite simple groups by a set of integers which is determined by two abstract parameter "taking order" and "being solvable". The origin of this consideration is in Thompson conjecture as follows.

Conjecture 1 (J.G. Thompson) Let G be a finite group with trivial center and S a non abelian simple group. $N(G)$ stands for the set of sizes of conjugacy classes. If $N(G)$ coincides with $N(S)$, then G is isomorphic to S.

Problem 1 Let G be a finite group and S a non abelian finite simple group. Let $M(G)$ stand for the set of orders of all elements of G. Assume

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1991 Mathematics Subject Classification : Primary 20D05, 20D06, 20D20.