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Volterra integral equations: the singular case

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Abstract. Positive solutions are established for the Volterra integral equation $y(t) = \int_0^t k(t,s)\, f(s, y(s))\, ds$, $t \in [0, T]$. Our nonlinearity may be singular at $y = 0$.

Key words: Volterra integral equation, singular, lower type inequalities, positive solution.

1. Introduction

This paper discusses the singular Volterra equation

$$y(t) = \int_0^t k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0, T], \quad T > 0 \ \text{ fixed.} \qquad (1.1)$$

Our nonlinearity f may not be a Carathéodory function because of the singular behavior of the y variable i.e. f may be singular at $y = 0$. In the literature (see [3, 4] and the references therein) almost all results concern the case when f is a L^∞-Carathéodory function; to our knowledge only one paper [1] has discussed (1.1), in its full generality, when f is singular at $y = 0$. We also note that only a handful of papers (see [2, Chapter 1]) have discussed the initial value problem (which is a special case of (1.1)),

$$\begin{cases} y^{(n)} = \phi(t)\, f(t, y) \quad \text{for} \quad t \in [0, T] \\ y^{(i)}(0) = 0, \quad 0 \le i \le n-1, \quad n \ge 1 \end{cases}$$

when f is singular at $y = 0$. This paper presents new results for (1.1). In particular new "lower type inequalities" on solutions to (1.1) are presented. Also by exploiting the monotonicity of the kernel we are able to relax some of the assumptions in [1]. For example if we consider the initial value problem

$$\begin{cases} y'' = [y(t)]^{-a} + A[y(t)]^b \quad \text{for} \quad t \in [0, T] \\ y(0) = y'(0) = 0, \quad A > 0, \quad 0 \le b \le 1, \quad a > 0, \end{cases} \qquad (1.2)$$

then the results in [1] guarantee that (1.2) has a solution if $a \in \left(0, \frac{1}{2}\right)$ whereas the results in this paper guarantee that (1.2) has a solution if

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