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# Extrinsic upper bounds for the first eigenvalue of elliptic operators

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**Abstract.** We consider operators defined on a Riemannian manifold $M^m$ by $L_T(u) = -\mathbf{div}(T\nabla u)$ where $T$ is a positive definite symmetric (1, 1)-tensor such that $\mathbf{div}(T) = 0$. We give an upper bound for the first nonzero eigenvalue $\lambda_{1,T}$ of $L_T$ in terms of the second fundamental form of an immersion $\phi$ of $M^m$ into a Riemannian manifold of sectional curvature bounded above by $\delta$. We apply these results to a particular family of operators defined on hypersurfaces of the space forms and we prove a stability result.

*Key words*: $r$-th mean curvature, Reilly's inequality.

## 1. Introduction

Let $(M^m, g)$ be a compact, connected $m$-dimensional Riemannian manifold. In this paper, we are interested in extrinsic upper bounds for the first nonzero eigenvalue of elliptic operators defined on $(M^m, g)$ (i.e. in terms of the second fundamental form of an isometric immersion of $(M^m, g)$ into an $n$-dimensional Riemannian manifold $(N^n, h)$). The elliptic second order differential operators $L_T$, which we are interested in, are of the form

$$L_T u = -\mathbf{div}_M(T\nabla^M u), \quad u \in C^\infty(M),$$

where $T$ is a $(1, 1)$-tensor on $M$ (which will be divergence-free and symmetric), and $\mathbf{div}_M$ and $\nabla^M$ denote respectively the divergence and the gradient with respect to the metric $g$. In the sequel, we will denote by $\lambda_{1,T}$, the first nonzero eigenvalue of such operator $L_T$.

When $T$ is the identity, $L_T = L_{\mathrm{Id}}$ is nothing but the Laplace operator of $(M^m, g)$. In this case, it is well known that if $(M^m, g)$ is isometrically immersed in the simply connected space form $N^n(c)$ ($c = 0, 1, -1$ respectively for the Euclidean space $\mathbb{R}^n$, the sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$), then we have the following estimate of $\lambda_1 = \lambda_{1,\mathrm{Id}}$ in terms of the square of the length of the mean curvature

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