On minimal points of Riemann Surfaces, I.

Dedicated to Prof. Yoshie Katsurada on her 60th birthday

By Zenjiro KURAMOCHI

Let R be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0,1,2,\cdots)$ be its exhaustion with compact analytic relative boundary ∂R_n. Let G be a domain in R such that the relative boundary ∂G consists of at most an enumerable number of smooth curves clustering nowhere in R. In the present paper we consider only domains as mentioned above. If an open set G has a smooth relative boundary, we call it a regularly open set. Let G_1 be a domain in $R-R_0$ and G_2 be a regularly open set in G_1 such that $G_1 \supset \bar{G}_2$, where $\bar{G}_2$ is the closure of G_2 in R. Let $U_n(z)$ be a harmonic function in $(G_1-\bar{G}_2)\cap R_n$ such that $U_n(z)=1$ on $\partial G_2 \cap R_n$, $=0$ on $\partial G_1 \cap R_n$ and $\frac{\partial}{\partial n} U_n(z)=0$ on $\partial R_n \cap (G_1-\bar{G}_2)$. If the Dirichlet integral $D(U_n(z))<M<\infty$ for any n, then $U_n(z)$ converges locally uniformly and in Dirichlet integral as $n\to\infty$ to a harmonic function which is denoted by $\omega(\bar{G}_2, z, G_1)$ and is called the Capacitary Potential[1] of $\bar{G}_2$ relative to G_1 (abbreviated by C. P.). Clearly $\omega(\bar{G}_2, z, G_1)$ is uniquely determined and has minimal Dirichlet integral (M. D. I.) over $(G_1-\bar{G}_2)$ among all harmonic functions with the same value as $\omega(\bar{G}_2, z, G_1)$ on $\partial G_1+\partial G_2$. Let $G_2\supset G_3\supset G_4, \cdots$ be a decreasing sequence of regularly open sets. Then $\omega(\bar{G}_n, z, G_1)$ converges locally uniformly and in Dirichlet integral as $n\to\infty$ to a harmonic function denoted by $\omega(\{\bar{G}_n\}, z, G_1)$. This is called C.P. of $\{\bar{G}_n\}$. Len F be a closed set in G_1, if there exists a harmonic function $U(z)$ in G_1-F such that $U(z)=0$ on ∂G_1, $U(z)=1$ on F and $D(U(z))<\infty$, there exists a uniquely determined harmonic function $\omega(F, z, G_1)$ such that $\omega(F, z, G)=1$ on ∂F except at most a set of capacity zero, $=0$ on ∂G_1 and has M. D. I. Let $\{F_n\}$ be a decreasing sequence in G_1. Then we can define C.P. of $\{F_n\}$ as above. Let $W(z)$ be the least positive harmonic function in G_1-F such that $W(z)=1$ on ∂F except at most a set of capacity zero. We call $W(z)$ the harmonic measure (abbreviated by H. M.) and denote it by $W(F, z, G_1)$. Similarly H. M. of a decreasing sequence $\{F_n\}$ can be defined also.

Let G be a domain in R such that $\bar{G}\cap \bar{R}_0=0$ and let $N_n(z,p)$ be a positive harmonic function in $(G-\{p\})\cap R_n$: $p\in R_n\cap G$ such that $N_n(z,p)=0$ on $\partial G\cap R_n$, $N_n(z,p)$ has a logarithmic singularity with coefficient 1 at p and