On direct modules

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

By Tsutomu TAKEUCHI

Y. Utumi obtained that if a ring R is left self-injective then so is the residue class ring R/J modulo the Jacobsn radical J of R. And B. L. Osofsky [5] extended this result to the case of endomorphism rings of quasi-injective modules. In this note we study endomorphism rings of those modules which are weaker than quasi-injectives, conforming to the method by Utumi [8].

1. Preliminaries. We will assume throughout that R is a nonzero ring with identity and that $M={}_RM$ denotes a nonzero unital left R-module. Let ${}_RA$ be an (R-)submodule of ${}_RM$. A complement ${}_RA^c$ of ${}_RA$ in ${}_RM$ is a maximal submodule of ${}_RM$ such that $A\cap A^c=0$. And, a double complement ${}_RA^{cc}$ of ${}_RA$ in ${}_RM$ is a complement of a complement of ${}_RA$ in ${}_RM$ such that $A\subset A^{cc}$. Zorn's lemma ensures the existence of ${}_RA^c$ and ${}_RA^{cc}$ for every submodule ${}_RA$ of ${}_RM$. ${}_RA$ is called complemented in ${}_RM$ if ${}_RA$ is a complement of some submodule of ${}_RM$ in ${}_RM$. To be easily seen, every direct summand of ${}_RM$ is complemented in ${}_RM$. Moreover, ${}_RA$ is essential in ${}_RA^{cc}$ and ${}_RA^{cc}$ is (essentially) closed in ${}_RM$, i.e., ${}_RA^{cc}$ has no proper essential extension in ${}_RM$.

The above leads the following smoothly:

LEMMA 1. *Let ${}_RA$ be a submodule of ${}_RM$. Then the following conditions are equivalent:*

(i) *${}_RA$ is closed in ${}_RM$.*

(ii) *${}_RA$ is complemented in ${}_RM$.*

(iii) *$A=A^{cc}$ for some double complement ${}_RA^{cc}$ of ${}_RA$ in ${}_RM$.*

(iv) *$A=A^{cc}$ for every double complement ${}_RA^{cc}$ of ${}_RA$ in ${}_RM$.*

(v) *Let ${}_RB$ be any submodule of ${}_RM$ contained in A. If ${}_RB$ is essential in ${}_RA$, then there exists such a double complement ${}_RB^{cc}$ of ${}_RB$ in ${}_RM$ that $B^{cc}=A$.*

The following notations will be adopted henceforth. Let ${}_RM$ be a left R-module and let S be the (R-)endomorphism ring of ${}_RM$, acting on the right side. Therefore $M={}_RM_S$ is a left R- and right S-bimodule. For ${}_RM$ we set

$$Z({}_RM)=\{a\in M\,|\,{}_R^Ra \text{ is essential in } {}_RR\},$$
$$Z(M_S)=\{a\in M\,|\,a^S_S \text{ is essential in } S_S\}$$