STRONGLY π-REGULAR RINGS

By

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ARENS-KAPLANSKY [1] and KAPLANSKY [3] investigated, as generalizations of algebraic algebras and rings with minimum condition, following two types of rings: one is a π-regular ring, that is, a ring in which for every element a there exists an element x and a positive integer n such that $a^n x a^n = a^n$, and the other is a ring in which for every a there exists an x and an n such that $a^{n+1}x = a^n$ — this we shall call a right π-regular ring. The present note is devoted mainly to study the latter more precisely. Apparently, the two notions of π-regularity and right π-regularity are different ones in general. However we can prove, among others, that under the assumption that a ring is of bounded index (of nilpotency) it is π-regular if and only if it is right π-regular. Moreover, we shall show, in this case, that we may find, for every a, an element z such that $az = za$ and $a^{n+1}z = a^n$, where n is the least upper bound of all indices of nilpotency in the ring. This is obviously a stronger result than a theorem of KAPLANSKY (2) as well as that of GERTSCHIKOFF (3), both of which are stated in section 8 of KAPLANSKY [3].

1. Strong regularity. Let A be a ring. Let a be an element of A. a is called *regular* (in A) if there exists an element x of A such that $axa = a$, while a is said to be *right* (or *left*) *regular* if there exists x such that $a^2x = a$ (or $xa^2 = a$). Further, we call a *strongly regular* if it is both right regular and left regular.

Lemma 1. *Let a be a strongly regular element of A. Then there exists one and only one element z such that $az = za$, $a^2z (= za^2) = a$ and $az^2 (= z^2a) = z$, and in particular a is regular. For any element x such that $a^2x = a$, z coincides with ax^2. Moreover, z commutes with every element which is commutative with a.*

Proof. Let x, y be two elements such that $a^2x = a$, $ya^2 = a$. Then

$$(1) \qquad ax = ya^2x = ya,$$

so that