ON THE RATIOS OF THE NORMS DEFINED BY MODULARS

By

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§ 1. Let R be a *modulared semi-ordered linear space* and $m(x)$ $(x \in R)$ be a *modular*[1] on R. Since $0 \leq m(\xi x)$ is a non-trivial convex function of real number $\xi \geq 0$ for every $0 \neq x \in R$, we can define two kinds of norms by the modular m as follows:

$$(1.1) \qquad \| x \| = \inf_{\xi > 0} \frac{1+m(\xi x)}{\xi}, \quad |\!|\!| x |\!|\!| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|} \qquad (x \in R).$$

The former of them is said to be the *first norm* by m and the latter to be the *second* (or *modular*) *norm* by m.

Let $\overline{R}^m$ be the *modular conjugate space* of R and $\overline{m}$ be the *conjugate modular*[2] of m. Then we can also define the norms on $\overline{R}^m$ by $\overline{m}$ as above. It is well-known [4; § 40] that *if R is semi-regular*[3] *the first norm by the conjugate modular $\overline{m}$ is the conjugate one of the second norm by m and the second norm by $\overline{m}$ is the conjugate one of the first norm by m.* Since $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are semi-continuous, they are reflexive [3]. We have always $|\!|\!| x |\!|\!| \leq \| x \| \leq 2 |\!|\!| x |\!|\!|$ for all $x \in R$, that is, $1 \leq \frac{\| x \|}{|\!|\!| x |\!|\!|} \leq 2$ for all $0 \neq x \in R$.

When the ratios of these two norms are equal to a constant number, i.e. $\frac{\| x \|}{|\!|\!| x |\!|\!|} = \gamma$ holds for each $0 \neq x \in R$, S. Yamamuro [8] and I. Amemiya [1] succeeded in showing that the modular m is of L^p-type essentially, i.e. $m(\xi x) = \xi^p m(x)$ for all $x \in R$ and $\xi \geq 0$, where $1 \leq p$.

In the earlier paper [7] the author investigated the case that the

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1) For the definition of a modular see [4]. The notations and terminologies used here are the same as in [4 or 7].

2) $\overline{R}^m$ is the totality of all linear functionals $\overline{a}$ on R such that $\inf_{\lambda \in \Lambda} |\overline{a}(x_\lambda)| = 0$ for every $x_\lambda \downarrow_{\lambda \in \Lambda} 0$ and $\sup_{m(x) \leq 1} |\overline{a}(x)| < +\infty$. The conjugate modular $\overline{m}$ of m on $\overline{R}^m$ is defined as

$$\overline{m}(\overline{a}) = \sup_{x \in R} \{\overline{a}(x) - m(x)\} \quad (\overline{a} \in \overline{R}^m).$$

3) R is said to be *semi-regular*, if $\overline{a}(x) = 0$ for all $\overline{a} \in \overline{R}^m$ implies $x = 0$.