# A NOTE ON NON-COMMUTATIVE KUMMER EXTENSIONS

By

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Let a simple ring $A$ (with 1 and minimum condition) be strictly Galois with respect to (an $F$-group) $\mathfrak{H}$ in the sense of [2]. Then $B=J(\mathfrak{H}, A)$ is a simple ring with $[A:B]=\#\mathfrak{H}$, and the following facts have been given in [2] and [3]. (As to notations and terminologies used in this note, we follow [2].)

1°. Let $\mathfrak{N}$ be an $F$-subgroup of $\mathfrak{H}$. If $N=J(\mathfrak{N}, A)$, then $A/N$ is strictly Galois with respect to $\mathfrak{N}$, $[N:B]=(\mathfrak{H}:\mathfrak{N})$ and $\mathfrak{H}(N)=\mathfrak{N}$. In particular, if $\mathfrak{N}$ is an invariant subgroup of $\mathfrak{H}$ then $\mathfrak{H}|N\cong\mathfrak{H}/\mathfrak{N}$.

2°. $A$ contains an $\mathfrak{H}$-normal basis element ($\mathfrak{H}$-n.b.e.), that is, $A$ contains an element $a$ such that $\{a\sigma; \sigma\in\mathfrak{H}\}$ forms a (linearly independent) right $B$-basis of $A$.

3°. If $\sigma\to x_\sigma$ is an anti-homomorphism of $\mathfrak{H}$ into $B^{\cdot}$ (the multiplicative group of units of $B$) then there exists an element $x\in A^{\cdot}$ such that $x\sigma=xx_\sigma$.

4°. Let $\mathfrak{H}$ be cyclic with a generator $\sigma$ of order $m$, and $B\cap C$ ($C$ the center of $A$) contains a primitive $m$-th root of 1. If there exists an element $a\in A^{\cdot}$ such that $a\sigma=a\zeta$, there holds $A=\oplus_{i=0}^{m-1}Ba^i=\oplus_{i=0}^{m-1}a^iB$.

Further, $A/B$ was called an $\mathfrak{H}$-Kummer extension if $\mathfrak{H}$ is a commutative $DF$-group whose exponent is $m_0$ and $B\cap C$ contains a primitive $m_0$-th root of 1, and [3, Theorem 3] enabled us the notion of an $\mathfrak{H}$-Kummer extension to be naturally regarded as a generalization of the classical one for (commutative) fields. On the other hand, in his paper [1], C. C. Faith proved that any commutative Kummer extension $A/B$ is completely basic, more precisely, every normal basis element of $A/B$ is a normal basis element of $A/B'$ for any intermediate field $B'$ of $A/B$. The purpose of this note is to carry over the last proposition to division rings. In fact, by the validity of $1°-4°$, a slight modification of Faith's proof will accomplish our attempt. Firstly, we exhibit the following characterization of an $\mathfrak{H}$-Kummer extension.

**Theorem 1.** *Let $\mathfrak{H}=\{\eta_1, \cdots, \eta_m\}$ be a DF-group of $A$ whose exponent is $m_0$. If $A/B$ is an $\mathfrak{H}$-Kummer extension then $A=\oplus_{i=1}^{m}a_iB=\oplus_{i=1}^{m}Ba_i$ with some $a_i\in A^{\cdot}$ such that every $\zeta_{ij}=a_i^{-1}\cdot a_i\eta_j$ is contained in $B\cap C$, and conversely.*