ON AN EQUIVALENCE RELATION ON SEMI-ORDERED LINEAR SPACES

By

Tetsuya SHIMOGAKI

§ 1. Let (E, Ω, μ) be a finite measure space with a countably additive non-negative measure μ defined on a σ-field Ω. Two real-valued μ-measurable functions $f(t)$ and $g(t)$ on E are called *mutually equi-measurable* [14], if $\mu\{t; f(t)>r\} = \mu\{t; g(t)>r\}$ holds for each real number r. If we write $f\sim g$, when f and g are mutually equi-measurable, it is observed easily that the relation $\sim$ is an equivalence relation on the space $\mathfrak{M}$ of all measurable functions on E. As is shown in [14], the concept of equi-measurability plays an important rôle in the theory of functions of real variables. Now let $\boldsymbol{X}$ be a linear space consisting of real-valued measurable functions, which is *semi-normal* in the sense of Nakano [11], i.e.

$$(1.1) \qquad 0\leqq f\in \boldsymbol{X}, \quad |g|\leqq f, \quad g\in\mathfrak{M} \text{ implies } g\in\boldsymbol{X},$$

where $0\leqq f$ means that $0\leqq f(t)$ holds almost everywhere. Evidently the function space $\boldsymbol{X}$ is considered as a universally continuous semi-ordered linear space[1] by this order.

We say that a function space $\boldsymbol{X}$ has the *weak rearrangement invariant property* (*w-RIP*), if $f\in\boldsymbol{X}$, $f\sim g$ always implies $g\in\boldsymbol{X}$, i.e. $\boldsymbol{X}$ is closed under the relation defined by equi-measurability. In the sequel, a function space $\boldsymbol{X}$ on E is termed to be a *Banach function space*[2] on E, if it is semi-normal and has a complete norm satisfying

$$(1.2) \qquad \|f\| = \sup_{\lambda\in\Lambda}\|f_\lambda\|, \quad \text{whenever } 0\leqq f_\lambda\uparrow_{\lambda\in\Lambda} f.$$

A Banach function space $\boldsymbol{X}$ is said to have the *strong rearrangement invariant property* (*s-RIP*), if $f\in X$, $f\sim g$ implies $g\in\boldsymbol{X}$ and $\|g\|\leqq A\|f\|$, where A is a fixed constant independent on f and g. $\boldsymbol{L}^p(E)$ spaces with $1\leqq p$, Orlicz spaces $\boldsymbol{L}_\Phi(E)$ and $\Lambda(\phi)$-spaces established by G. G. Lorentz [5, 6] and I. Halperin

---

1) A semi-ordered linear space R is called *universally continuous*, if $0\leqq a_\lambda$ $(\lambda\in\Lambda)$ implies $\bigcap_{\lambda\in\Lambda} a_\lambda\in R$, i.e. a conditionally complete vector lattice in Birkhoff's sense or a K-space in the sense of Vulich [12].

2) For the detailed properties of Banach function spaces see [7] or [13].