ON THE DISTRIBUTION OF INTEGERS REPRESENTABLE AS A SUM OF TWO h-TH POWERS

By

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Our aim in this note is to present some elementary results concerning the distribution of integers which can be expressed as a sum of two h-th powers, where $h \geqq 2$ is a fixed integer.

1. According to P. Erdös [2], R. P. Bambah and S. Chowla [1] have proved that for some *sufficiently large* constant C the interval $(n, n+Cn^{\frac{1}{4}})$ always contains an integer of the form x^2+y^2, n, x and y being integral, and Erdös [2] coniectures (among others) that this holds for every C if $n \geqq n_0(C)$. We cannot, at present, prove this conjecture of Erdös, but it is possible to refine the result of Bambah and Chowla in the following form:

Theorem 1. *For every $n \geqq 1$ there are integers x, y with $xy \neq 0$ satisfying*

$$n < x^2+y^2 < n+2^{\frac{3}{2}}n^{\frac{1}{4}} .$$

Proof. For $n=1$ and $n=2$ the result is obvious. Assume now that $n \geqq 3$.

Let δ, $0<\delta<1$, be a fixed real number: the exact value of δ (which may depend on n) will be determined in a moment later.

Write

$$[n^{\frac{1}{2}}] = n^{\frac{1}{2}}-(1-\varepsilon) \qquad (0<\varepsilon \leqq 1) .$$

Here, and in what follows, $[t]$ denotes, as usual, the greatest integer not exceeding t.

We distinguish two cases.

Case 1: $0<\varepsilon \leqq \delta$. We take

$$x=[n^{\frac{1}{2}}]+1 , \qquad y=1 .$$

Then we have

$$n < x^2+y^2 = n+2\varepsilon n^{\frac{1}{2}}+\varepsilon^2+1 < n+2^{\frac{3}{2}}n^{\frac{1}{4}} ,$$