ON HAAR FUNCTIONS IN THE SPACE $L_{M(\xi,t)}$

By

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1. It is well known [6, 8, 12] that Haar functions constitute a (Schauder) basis in Banach spaces $L^p[0,1]$ $(1 \leqq p < +\infty)$ and Orlicz spaces $L_M[0,1]$ with the Δ_2-condition. Generalizing this fact to an arbitrary separable Banach function space E on a measure space, H. W. Ellis and I. Halperin showed in [3] that Haar system of functions (in an extended sense) composes a basis in E, if a norm of E satisfies a condition called *levelling length property*[1]. Although this condition is sufficiently general, yet it is not always a necessary one.

In this note we shall show a sufficient condition in order that Haar functions be a basis for the Banach function space $L_{M(\xi,t)}[0,1]$ or $L^{p(t)}[0,1]$. In fact, we shall establish, as for the space $L^{p(t)}$, that *if $p(t)$ satisfies the Lipschitz α-condition* $(0<\alpha\leqq 1)$ *then Haar functions constitute a basis in* $L^{p(t)}$ (Theorem 4). As a matter of course, the norms of these spaces do not satisfy the above condition given in [3] except some special cases.

In **2** we shall introduce Haar functions, the function spaces $L_{M(\xi,t)}$ and $L^{p(t)}$[2] with the notations used here. The main theorems shall be stated in **3**, and some remarks shall be presented in **4**.

2. A sequence of functions defined on $[0,1]$: $\{\chi_\nu(t)\}_{\nu=1}^{\infty}$ is called a *system of Haar functions*, if $\chi_1(t)=1$ for all $t\in[0,1]$ and for $\nu=2^n+k$ $(n=0,1,2,\cdots;$ $k=1,2,\cdots,2^n)$[3]

$$(2.1)\qquad \chi_\nu(t)=\chi_{2^n+k}(t)=\begin{cases}\sqrt{2^n} & \text{for } t\in\left[\dfrac{2k-2}{2^{n+1}},\ \dfrac{2k-1}{2^{n+1}}\right),\\[2mm] -\sqrt{2^n} & \text{for } t\in\left(\dfrac{2k-1}{2^{n+1}},\ \dfrac{2k}{2^{n+1}}\right],\\[2mm] 0 & \text{otherwise in } [0,1].\end{cases}$$

1) A norm $\|\cdot\|$ of E is called to have the *levelling length property*, if $\|f_e\|\leqq\|f\|$ holds for any $f\in E$ and measurable set e, where f_e coincides with f outside the e and on e, $f_e=\left\{\dfrac{1}{d(e)}\displaystyle\int_e f(t)dt\right\}C_e$ (C_e is the characteristic function of e). This property was first discussed by them in the earlier paper [4]. At the same time, G. G. Lorentz and D. G. Wertheim also found it independently and named it the *average invariant property* [9].

2) In the sequel, we eliminate $[0,1]$ and write simply $L_{M(\xi,t)}$ (or $L^{p(t)}$) in place of $L_{M(\xi,t)}[0,1]$ (resp. $L^{p(t)}[0,1]$). $L^{p(t)}$ was first discussed by W. Orlicz in [11], and was investigated precisely by H. Nakano [10].

3) This formulation of Haar functions is due to Z. Ciesielskii [2].