A NOTE ON HOCHSCHILD COHOMOLOGY GROUPS

By

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Let K be a commutative ring with identity element, and let Λ be a K-algebra, that is, an algebra over K with identity element 1. We denote by Λ^* the opposite algebra of Λ, which is in an opposite-isomorphism $\lambda \to \lambda^*$ with Λ. Any right Λ-module is converted into a left Λ^*-module (and conversely) by setting the left multiplication of λ^* to be the right multiplication of λ. Furthermore, every two-sided Λ-module is, and in particular Λ is, regarded as a left module for the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^*$[1] in the natural manner.

Let A be a two-sided Λ-module. Hochschild defined in [2] the cohomology groups of Λ for A as the homology groups $H^n(\Lambda, A)$ of the complex whose components are the cochain groups $C^n(\Lambda, A)$, i.e., the module of all K-multilinear mappings $f = f(\lambda_1, \cdots, \lambda_n)$ of Λ into A, and whose differentiation operators δ^n: $C^n(\Lambda, A) \to C^{n+1}(\Lambda, A)$ are defined by

$$(\delta^n f)(\lambda_1, \cdots, \lambda_{n+1}) = \lambda_1 f(\lambda_2, \cdots, \lambda_{n+1}) + \sum_{i=1}^{n} (-1)^i f(\lambda_1, \cdots, \lambda_i \lambda_{i+1}, \cdots, \lambda_{n+1}) + (-1)^{n+1} f(\lambda, \cdots, \lambda_n) \lambda_{n+1} .$$

Here, the two-sided Λ-module A needs not be assumed to be unital (i.e., the identity element 1 of Λ does not necessarily act on A as the identity-operator on both left and right hands), but we may replace A by the unital module $1A1$ to obtain the same cohomology groups according to Hchschild [3], Th. 1 : $H^n(\Lambda, A) \cong H^n(\Lambda, 1A1)$. On the other hand, Cartan and Eilenberg gave in [1] another definition of cohomology groups for a unital two-sided Λ-module A; namely, they called $\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$ the n-th cohomology group of Λ for A. The defined two groups $H^n(\Lambda, A)$ and $\mathrm{Ext}^n_{\Lambda^e}(\Lambda, A)$, for unital A, coincide (that is, are naturally equivalent as functors of A) always for $n=0$, 1 ([1], Chap. IX, Prop. 4.1.). But this is not the case in general for $n>1$. It is shown in [1], Chap. IX, §6 that the both groups coincide if Λ is K-projective. In this note, we shall however generalize this to prove that the both groups coincide whenever Λ^e is projective as a right Λ-module. In this connection, it may be of some interest to give in Proposition 2 below a homological significance of the Hochschild two-sided Λ-module $\mathrm{Hom}_K(\Lambda, A)$ introduced in [2].

1) We shall mean by the mere $\otimes$ the tensor product over K; thus, $\Lambda \otimes \Lambda^* = \Lambda \otimes_K \Lambda^*$.