ON A THEOREM CONCERNING THE DISTRIBUTION OF ALMOST PRIMES

By

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By an *almost prime* is meant a positive rational integer the number of prime factors of which is bounded by a certain constant. Let us denote by $\Omega(n)$ the total number of prime factors of a positive integer n. In 1920 Viggo Brun [2] elaborated an elementary method of the sieve of Eratosthenes to prove that for all sufficiently large x there exists at least one integer n with $\Omega(n) \leqq 11$ in the interval $x \leqq n \leqq x + x^{\frac{1}{2}}$. Quite recently W. E. Mientka [4] improved this result of Brun, showing that for all large x there exists at least one integer n with $\Omega(n) \leqq 9$ in the interval $x \leqq n \leqq x + x^{\frac{1}{2}}$. To establish this Mientka makes use of the sieve method due to A. Selberg instead of Brun's method (cf. [3] and [4]). By refining the argument of Mientka [4] we can further improve his result. Indeed, we shall prove in this paper the following

Theorem. *Let $k \geqq 2$ be a fixed integer. Then, for all sufficiently large x, there exists at least one integer n with $\Omega(n) \leqq 2k$ in the interval $x < n \leqq x + x^{1/k}$.*

Thus, in particular, if $k=2$ then for all large x the interval $x < n \leqq x + x^{\frac{1}{2}}$ always contains an integer n such that $\Omega(n) \leqq 4$. Of course, the restriction in the theorem that k be integral may be relaxed without essential changes in the result.

Let us mention that the existence of a prime number p in the interval $x < p \leqq x + x^{1/k}$ for all large x could not be deduced, as is well known, even from the Riemann hypothesis if only $k=2$.

Note. It is possible to generalize our theorem presented above so as to concern with the distribution of almost primes in an arithmetic progression. Thus, if a and b are integers such that $a \geqq 1$, $0 \leqq b \leqq a-1$, $(a, b)=1$, then we can prove the existence of an integer n satisfying

$$x < n \leqq x + x^{1/k}, \quad n \equiv b \pmod{a},$$
$$\Omega(n) \leqq 2k,$$

provided that x be sufficiently large, $k \geqq 2$ being a fixed integer. Here, in particular, in the case of $k=2$, the inequality $\Omega(n) \leqq 4$ may be replaced by $\Omega(n) \leqq 3$: this result is apparently stronger than the above theorem for the