REMARKS ON COMPLETENESS OF CONTINUOUS FUNCTION LATTICE

By

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Let E be an arbitrary topological space and $C(E)$ be a vector lattice of all real valued continuous functions on E. In general the lattice $C(E)$ is neither conditionally complete[1] nor conditionally σ-complete[2]. H. Nakano shows in [1] that a sufficient condition for $C(E)$ to be conditionally σ-complete (conditionally complete) is that E is *σ-universal* (*universal*), that is, every open F_σ-set has an open closure (every open set has an open closure) (cf. [2] Chap. VII, Theorem 41.1, Theorem 41.4). Under the assumption that E is normal (completely regular) σ-universality (universality) of E is a necessary condition for $C(E)$ to be conditionally σ-complete (conditionally complete). L. Gillman and M. Jerison in their book [3] show that for a completely regular space E the necessary and sufficient condition for $C(E)$ to be conditionally σ-complete is that E is *basically disconnected*, that is, every cozero-set[3] has an open closure ([3] p. 51, 3N). In this note we shall remark the necessary and sufficient topological condition for $C(E)$ on an arbitrary topological space E to be conditionally σ-complete or conditionally complete.

In the sequel a cozero-set P of $f\in C(E)$ will be denoted by $P(f)$; $P(f) = \{x|f(x)\neq 0\} = \{x||f|(x)>0\}$.

Theorem 1. *$C(E)$ is a conditionally σ-complete lattice if and only if the following two conditions are satisfied*

a) *there exists the smallest open-closed set $U(P)$ containing P for any cozero-set P.*

b) *if $P_1\frown P_2=\phi$ for two cozero-sets P_1 and P_2, then $U(P_1)\frown U(P_2)=\phi$.*

Proof. Suppose $C(E)$ is conditionally σ-complete and P is a cozero-set of some $f\in C(E)$, $P=P(f)$, then by the conditional σ-completeness of $C(E)$ f gives the orthogonal decomposition of the constant function $\mathbf{1}$ as follows

$$\mathbf{1} = [f]\mathbf{1} + [f]^{\perp}\mathbf{1}$$

1) every family with an upper bound in $C(E)$ has a supremum in $C(E)$.

2) every countable family with an upper bound in $C(E)$ has a supremum in $C(E)$.

3) $\{x|f(x)=0\}$ is a zero-set of $f\in C(E)$, cozero-set is a complement of a zero-set.