On perturbation of closed operators in a Banach space

Didicated to Professor Yoshie Katsurada on her 60th birthday

By Atsushi YOSHIKAWA*)

Introduction.

This paper deals with the perturbation problems of closed linear operators A and B in a Banach space. We give in §1 an elementary criterion in order that $A+B$ be again closed (See Theorem 1.2). We apply this result to perturbation problems of two accretive operators A and B and obtain a criterion in order that $-(A+B)$ generate a strongly continuous semi-group of contraction operators (See Theorem 2.7). However, we should note that the essence of our criterion was discovered by Trotter. In §3 we develop a Hilbert space theory and obtain several sufficient conditions, covering Nelson's condition (See Theorems 3.7 and 3.10).

§1. A criterion for closedness of $A+B$.

Consider two linear operators A and B in a Banach space X. We define a third operator $A+B$ by

(1.1) $$(A+B)x = Ax + Bx \quad \text{for} \quad x \in \mathbf{D}(A+B) = \mathbf{D}(A) \cap \mathbf{D}(B).$$ [1]

We exclude from our consideration the trivial case $\mathbf{D}(A) \cap \mathbf{D}(B) = 0$.

Now we pose the following

PROBLEM 1.1. Assume that A and B be closed. When is $A+B$ also closed?

The following result is our partial answer to this problem.

THEOREM 1.2. *Assume that the resolvent set of A, $\mathbf{P}(A)$, be non-empty and that there be a $\lambda \in \mathbf{C}$ such that $\lambda + B$ is of closed range and invertible. Let $-\mu \in \mathbf{P}(A)$. Then the following two conditions are equivalent*:

(1.2) $$A+B \text{ is closed and } -\mu \in \mathbf{P}(A+B);$$

(1.3) $$-1 \in \mathbf{P}\big(B(\mu+A)^{-1}\big).$$

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1) $\mathbf{D}(T)$ stands for the definition domain of a linear operator T in X. We shall denote by $\mathbf{R}(T)$ the range of T, and by $\mathbf{N}(T)$ the null-space of T.