A remark on doubly transitive groups

To Professor Yoshie Katsurada on the occasion of her 60th birthday

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1. This note is a continuation of [12]. We shall use the same notation. The purpose of this note is to prove the following.

THEOREM. *Let $\mathfrak{G}$ be a doubly transitive permutation group of odd degree satisfying the following conditions.*

(1) $\mathfrak{G}_{1,2}$ *is of even order,*

(2) *All Sylow subgroups of* $\mathfrak{G}_{1,2}$ *are cyclic,*

(3) $\chi(\tau)$ *contains a regular normal subgroup,*

(4) $\mathfrak{G}$ *has one class of involutions,*

(5) $\mathfrak{G}_{1,2}$ *has unique involution.*

Then $\mathfrak{G}$ *contains a regular normal subgroup.*

From this and [12, Theorem] we obtain the following.

COROLLARY. *Let $\mathfrak{G}$ be a doubly transitive permutation group of odd degree satisfying the above conditions* (1), (2) *and* (3). *Then $\mathfrak{G}$ contains a regular normal subgroup or it is isomorphic to one of the groups S_5 with $n=5$ and* PSL (2, 11) *with $n=11$.*

2. Assume $\mathfrak{G}$ does not contain a regular normal subgroup. By [12, Theorem 1] we may assume that $|\mathfrak{K}|>2$ and $\mathfrak{K}_0=\langle\tau\rangle$. Thus $d/2$ is odd. From the condition (4) a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ is also a Sylow 2-subgroup of $\mathfrak{G}$.

LEMMA 1. *A Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ is not metacyclic.*

PROOF. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ containing $\langle\mathfrak{K}, I\rangle$ and let $\mathfrak{S}'$ be a cyclic normal subgroup of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{S}'$ is cyclic. If $|\mathfrak{S}/\mathfrak{S}'|>2$, then $\mathfrak{G}$ is solvable by [11]. Therefore $\mathfrak{S}=\langle I, \mathfrak{S}'\rangle$. Since $\mathfrak{K}\neq\langle\tau\rangle$, $|\mathfrak{S}'|>2$. If $\mathfrak{S}$ is abelian, then $\mathfrak{G}$ is solvable by the Burnside's splitting theorem. If $\mathfrak{S}$ is dihedral or semi-dihedral, then $\mathfrak{K}_0\neq\langle\tau\rangle$, which is a contradiction. If $S^I=S\tau$ for a generator S of $\mathfrak{S}'$, $\mathfrak{G}$ is solvable by [13]. Thus $\mathfrak{S}$ is not metacylic.

LEMMA 2. *$\chi_1(\tau)$ is contained in $C_{\mathfrak{G}}(I)$.*

PROOF. Assume that there exists a Sylow q-subgroup $\mathfrak{H}_q'$ of $\chi_1(\tau)$ such that $\langle\mathfrak{H}_q', I\rangle$ is dihedral. Let $\mathfrak{S}'$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\mathfrak{H}_q')$ containing