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An example of a globally hypo-elliptic operator

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§ 1. Introduction

Let $T^2=R^2/2\pi\mathbf{Z}^2$ be the 2-dimmensional torus. A function $f(x, y)$ of $(x, y)\in R^2$ is identified with a function on the torus $\mathbf{T}^2$ if and only if it is doubly periodic, i. e.,

(1) $\qquad f(x+2n\pi,\ y+2m\pi)=f(x, y)$ for any n and m in $\mathbf{Z}$.

We consider a linear partial differential operator of the second order

(2) $$L=-\frac{\partial^2}{\partial x^2}-\phi(x)^2\frac{\partial^2}{\partial y^2},$$

where $\phi(x)$ is a real-valued function of x of class C^∞. We assume that

(3) $$\phi(x)=1 \text{ for } |x|<\frac{\pi}{2},$$
$$=0 \text{ for } \frac{3}{4}\pi\leqq|x|\leqq\pi$$

and that $\phi(x)$ is periodic, i. e., $\phi(x)=\phi(x+2\pi)$.

The aim of this note is to show the following

THEOREM. *The operator L is hypo-elliptic. That is, if a distribution* $u\in\mathscr{D}'(\mathbf{T}^2)$ *satisfies*

(4) $\qquad Lu=f$

and if $f\in C^\infty(\mathbf{T}^2)$, *then* $u\in C^\infty(\mathbf{T}^2)$.

REMARK. Let U be an open set outside the support of the function $\phi(x)$. Then the restriction of L to U coincides with $-\left(\frac{\partial}{\partial x}\right)^2$. This means that the operator L is not locally hypo-elliptic. Let $X_1=\frac{\partial}{\partial x}$ and $X_2=\phi(x)\frac{\partial}{\partial y}$. Then these vector fields do not satisfy Fefferman-Phong condition [2]. However they are controlable in the sense of Amano [1].

§ 2. Proof.

We shall begin with the following