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On the variational problem associated with standard differential systems

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1. Preliminaries

1.1. Left invariant differential systems on Lie groups, and conditions $(C_0), (C)$ and (C')

At the outset we consider a new condition on differential systems (M, D) as follows:

(C_0) any two points p and g of M can be connected by a piece-wise regular integral curve of (M, D).

Clearly condition (C_0) implies condition (C). Suppose that M is connected, and $\mathrm{rank}(D) > 0$. Then it is known that condition (C') implies condition (C_0): (see Appendix in [5]). Furthermore it can be shown that if (M, D) is real analytic, the three conditions $(C_0), (C)$ and (C') are mutually equivalent. The proof of this fact is based on the fact above and Nagano's theorem ([4]) on real analytic involutive differential systems possibly with singularities.

Now, let G be a connected Lie groups and $\mathfrak{g}$ its Lie algebra. In the present paper $\mathfrak{g}$ is defined to be the tangent space $T_e(G)$, e being the identity element $\mathfrak{g}$, equipped with the natural Lie algebra structure. We denote by exp the exponential mapping of $\mathfrak{g}$ to G, and by L_a the left translation of G corresponding to an element of G.

Let ϑ be a subspace of $\mathfrak{g}$. We denote by D the left invariant differential system on G induced by the subspace ϑ: (i) $\mathrm{D_e} = \vartheta$, and (ii) $D_a b = dL_a(D_b)$, $a, b \in G$. It is clear that the differential system (M, D) satisfies condition (C'), if and only if the Lie algebra $\mathfrak{g}$ is generated by ϑ. For completeness we shall prove the following

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Professor emeritus Noboru Tanaka has passed away on March 4 in 2011. The present manuscript was completed after his retirement from Hokkaido University and later communicated to one of the editors (Keizo Yamaguchi) by his family. We add the table of contents, some notes and references to the original manuscript. – Editor's note – May 20, 2013.