Integrability conditions for almost quaternion structures

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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§ 0. Introduction

Suppose that there are given, in a differentiable manifold, 3 tensor fields F, G and H of type (1, 1) satisfying

$$F^2=-1,\quad G^2=-1,\quad H^2=-1,$$
$$F=GH=-HG,\quad G=HF=-FH,\quad H=FG=-GF.$$

We then call the set (F, G, H) an *almost quaternion structure* and the manifold an *almost quaternion manifold.*

If we can cover the manifold by a system of coordinate neighborhoods with respect to which components of F, G and H are all constant, we say that the almost quaternion structure (F, G, H) is *integrable* and call the structure a *quaernion structure.*

The integrability conditions for almost quaternion structures and the existence of an affine connection with respect to which F, G and H are all parallel have been studied by Bonan [1] and Obata [4], [5].

The main purpose of the present paper is to discuss these problems making use not only of the Nijenhuis tensors $[F, F]$, $[G, G]$, $[H, H]$ but also of the Nijenhuis tensors $[G, H]$, $[H, F]$, $[F, G]$.

§ 1. Preliminaries

Let P and Q be two tensor fields of type (1, 1) in a differentiable manifold. It is well known (Kobayashi and Nomizu [3]) that the expression given by

$$\begin{aligned}(1.1)\qquad & [P, Q]\,(X, Y)\\ &=[PX, QY]-P[QX, Y]-Q[X, PY]\\ &\quad+[QX, PY]-Q[PX, Y]-P[X, QY]+(PQ+QP)[X, Y],\end{aligned}$$

X and Y being arbitrary vector fields, defines a tensor field of type (1, 2) and is called the Nijenhuis tensor of P and Q. If $P=Q$, we have

$$(1.2)\qquad [P, P]\,(X, Y)$$