On almost complex structures on the products and connected sums of the quaternion projective spaces

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1. Introduction and results

It is known by F. Hirzebruch [3] and by T. Heaps [2]*) that the quaternion projective space $P_n(Q)$ of quaternion dimension n has no almost complex structure for $n \neq 3$.

In this note, we consider whether almost complex structures exist or not on the product spaces $P_{n_1}(Q) \times \cdots \times P_{n_r}(Q)$ of quaternion projective spaces, and the connected sums

$$\alpha P_n(Q) \# \left(-\beta P_n(Q)\right) = \underbrace{P_n(Q) \# \cdots \# P_n(Q)}_{\alpha \text{ copies}} \# \underbrace{(-P_n(Q)) \# \cdots \# (-P_n(Q))}_{\beta \text{ copies}},$$

where the sign - denotes the reversed orientation.

THEOREM A. *The product spaces* $P_{n_1}(Q) \times \cdots \times P_{n_r}(Q)$ *for* $r \geqq 2$ *admit no almost complex structures if* $n_i \neq 2$, 3 *for an integer* i, $1 \leqq i \leqq r$.

THEOREM B. *The connected sums*

$$\alpha P_n(Q) \# \left(-\beta P_n(Q)\right), \quad \textit{where} \ \ n \leqq 10,$$

admit no almost complex structures if $n=1, 2, 4, 5, \cdots, 10$ *or* $n=3$, $\alpha \neq 3\beta+1$.

2. The product spaces $P_{n_1}(Q) \times \cdots \times P_{n_r}(Q)$.

Let $P_n(Q)$ be the quaternion projective space of quaternion dimension n. Let $p_i \in H^{4i}(P_n(Q); Z)$ be tthe ith Pontrjagin class. Let $c_i \in H^{2i}(P_n(Q); Z)$ be the ith Chern class if $P_n(Q)$ has an almost complex structure. Let $u \in H^4(P_n(Q); Z)$ be the canonical generator. By F. Hirzebruch [3] or A. Borel and F. Hirzebruch [1], we have the total Pontrjagin class of $P_n(Q)$,

$$p = \sum_{i=0}^{\infty} p_i = (1+u)^{2n+2}(1+4u)^{-1}.$$

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