A remark on 2-transitive groups of odd degree

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Let G be a 2-transitive group on $\Omega=\{1, 2, \cdots, n\}$, n odd. Let $G_{a,b}$ be the stabilizer of the points a, b and $g_1^*(2)$ the number of involutions in G_1 which fix only the point 1. Let $F(H)$ denote the set of all points fixed by a subset H of G and $\alpha(H)$ the number of points in $F(H)$. In this note we shall prove the following.

THEOREM *If* $|G_{1,2}|$ *is even and* $\alpha(G_{1,2})$ *is odd, then* $g_1^*(2)=1$ *and* G *has a regular normal subgroup or every involution of* G *is conjugate to an involution of* $G_{1,2}$.

PROOF. Let I be an involution of G with the cycle structure $(1, 2)\cdots$. Then I normalizes $G_{1,2}$. Let d be the number of elements of $G_{1,2}$ inverted by I. Then d is the number of involutions with cycle structures $(1, 2)\cdots$. Let $g(2)$ and $g_1(2)$ denote the number of involutions in G and G_1, respectively. Since G is 2-transitive, $G=G_1+G_1IG_1$ and hence $g(2)=g_1(2)+d(n-1)$. $d-g_1^*(2)=\{(g(2)-g_1^*(2)n)-(g_1(2)-g_1^*(2))\}/(n-1)$ is the number of involutions with the cycle structures $(1, 2)\cdots$ which are conjugate to an involution of $G_{1,2}$. Thus $g_1^*(2)$ is the number of involutions with cycle structures $(1, 2)\cdots$ which are not conjugate to any involution of $G_{1,2}$. Since $F(G_{1,2})^I=F(G_{1,2})$ and $\alpha(G_{1,2})$ is odd, $\alpha(\langle G_{1,2}, I\rangle)=1$. Let a be the point in $F(\langle G_{1,2}, I\rangle)$. Every involution in $IG_{1,2}$ fixes a. Assume $g_1^*(2)\neq 0$. Let L be the subgroup of G_a generated by $g_1^*(2)$ involutions with the cycle structures $(1, 2)\cdots$ which fix only a. Then L is characteristic in G_a and hence L is 1/2-transitive on $\Omega-\{a\}$. Since $\{1, 2\}^L=\{1, 2\}$, L is 2-group and every L-orbits in $\Omega-\{a\}$ is of length 2. If $g^*(2)\geq 2$, then there exists a L-orbit of length >2. Thus $g^*(2)=1$, and by Z^*-theorem $0(G)\neq 1$ and G has a regular normal subgroup. This proves Theorem.

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References

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