Notes on relatively harmonic immersions

By Shigeru ISHIHARA and Susumu ISHIKAWA

The notion of harmonic mappings was introduced and such mappings were studied by Eells and Sampson [1]. Recently, such mappings have been discussed by several authors (See [1], [2], [3], [4] and [5], for example) and many interesting results have been obtained. Yano and one of the present authors [5] have proved, concerning harmonic mappings, some theorems in which sufficient conditions for a harmonic mapping to be affine or homothetic are stated. To prove these theorems, they computed Laplacian $\Delta\|df\|^2$ of the square of the differential mapping df for a harmonic mapping f of a compact Riemannian space (M, g) into a Riemannian space $(N, \bar{g})$ and pinched in a certain sense the sum of eigenvalues of the tensor g^* induced in M from $\bar{g}$ by f. In the present paper, we define relatively harmonic immersions of a compact Riemannian space $(M, \bar{g})$ of dimension n into a Riemannian space $(N, \bar{g})$ of dimension $n+1$ (See § 1) and obtain some sufficient conditions for such an immersion to be relatively affine or homothetic by a similar way to that taken in [5]. The results will be stated in Theorems 4.1~4.5.

In § 1, notations and some concepts concerning immersions and relatively harmonic immersions will be defined and some propositions will be proved. In § 2 Laplacian $\Delta\|df\|^2$ will be computed and in § 3 some inequalities will be given for later use. The last § 4 is devoted to prove Theorems 4.1~4.5.

§1. Differentiable immersions of a Riemannian space into another

Let (M, g) and $(N, \bar{g})$ be two Riemannian spaces of dimension n and $n+1$ respectively, where $n \geqq 2$. Let there be given a differentiable immersion $f: M \to N$, that is, a differentiable mapping $f: M \to N$ whose rank is equal to n everywhere. Such an immersion will be sometimes denoted by $f:(M, g)\to(N, \bar{g})$. Manifolds, mappings and geometric objects we discuss are assumed to be differentiable and of class C^∞. Take a coordinate neighborhoods $\{U, x^h\}$ of M and $\{\bar{U}, y^\alpha\}$ of N in such a way that $f(U)\subset\bar{U}$, where local coordinates of M are denoted by $(x^h)=(x^1, \cdots x^n)$ and those of N by $(y^\alpha)=(y^{\bar{1}}, \cdots, y^{\overline{n+1}})$. The indices h, i, j, k, l, m, r, s run over the range $\{1, \cdots, n\}$ and the indices $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu$ over the range $\{\bar{1}, \cdots, \overline{n+1}\}$. The