Nonmaximal weak-*Dirichlet algebras

By Takahiko NAKAZI

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0. Introduction

Let A be a weak-*Dirichlet algebra of $L^{\infty}(m)$ which was introduced by Srinivasan and Wang [7]. Let $H^{\infty}(m)$ denote the weak-*closure of A in $L^{\infty}(m)$. Suppose there exists at least one positive nonconstant function v in $L^{1}(m)$ such that the measure vdm is multiplicative on A. Then Merrill [4] characterizes the classical space $H^{\infty}(d\theta)$ by invariant subspaces of $H^{\infty}(m)$ or the maximality of $H^{\infty}(m)$ as a weak-*closed subalgebra of $L^{\infty}(m)$. In section 1 we characterize $H^{\infty}(d\theta d\phi)$, which is certain weak-* Dirichlet algebra on the torus, by invariant subspaces of $H^{\infty}(m)$. We need not assume the existence of the above v. Then, in some special case, Muhly [6] shows that $H^{\infty}(m)$ is a maximal weak-*closed subalgebra of $L^{\infty}(m)$. But in general, $H^{\infty}(m)$ is not maximal and so there exist weak-* closed subalgebras of $L^{\infty}(m)$ which contain $H^{\infty}(m)$ properly. In section 2 we construct some typical subalgebra in such subalgebras and we determine forms of all weak-*closed subalgebras which contain this subalgebra. This is applied to determine forms of all subalgebras which contain $H^{\infty}(d\theta d\theta)$.

Recall that by definition a weak-*Dirichlet algebra is an algebra A of essentially bounded measurable functions on a probability measure space $(X, \mathfrak{M}, m)$ such that (i) the constant functions lie in A; (ii) $A+\bar{A}$ is weak-* dense in $L^{\infty}(m)$ (the bar denotes conjugation, here and always); (iii) for all f and g in A, $\int fg\,dm=(\int f dm)(\int g\,dm)$. The abstract Hardy spaces $H^{p}(m)$, $1\leq p\leq\infty$, associated with A are defined as follows. For $1\leq p<\infty$, $H^{p}(m)$ is the $L^{p}(m)$-closure of A, while $H^{\infty}(m)$ is defined to the weak-*closure of A in $L^{\infty}(m)$. For $1\leq p\leq\infty$, $H_{0}^{p}=\{f\in H^{p}(m): \int f dm=0\}$. For any subset $M\subseteq L^{\infty}(m)$, denote by $[M]_{2}$ the $L^{2}(m)$-closure of M. A closed subspace M of $L^{p}(m)$ is called B-invariant if $f\in M$ and $g\in B$ imply that $fg\in M$, where B is a subalgebra of $L^{\infty}(m)$. In particular, if $B=L^{\infty}(m)$, M is called doubly-invariant. For any measurable subset E of X, the function χ_{E} is the characteristic function of E. If $f\in L^{p}(m)$, write E_{f} for the support set of f and write χ_{f} for the characteristic function of E_{f}.

We use the following result.

(a) *If M is a weak-*closed A-invariant subspace of $L^{\infty}(m)$, then $M=$*