Positive approximants of normal operators

Takashi SEKIGUCHI

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1. Introduction. We consider the problem of approximation for a given bounded linear operator on a fixed Hilbert space by positive operators where positivety means non-negative semi-definite. Study of this problem was initiated by P. R. Halmos [4], who proved that the distance of an operator to the set of all positive operators is completely determined. The results proved by him can be formulated as follows.

Let A be a bounded linear operator on a Hilbert space $\mathscr{H}$. Put $A=B+iC$ where B and C denote the real part Re A and the imaginary part Im A of A respectively.

(1) *Put*

$$\delta=\inf\left\{\|A-P\|: P\geqq 0\right\}.$$

Then

$$\delta=\inf\left\{r\geqq 0: r^2\geqq C^2,\ B+(r^2-C^2)^{\frac{1}{2}}\geqq 0\right\}.$$

(2) *Define another norm* $|||\ \ |||$ *by*

$$|||A|||=\left\|(\mathrm{Re}\ A)^2+(\mathrm{Im}\ A)^2\right\|^{\frac{1}{2}}.$$

Then

$$\frac{1}{2}\|A\|\leqq|||A|||\leqq\|A\|$$

and

$$\delta=\inf\left\{|||A-P|||: P\geqq 0\right\}.$$

(3) *Put*

$$\mathscr{P}(A)=\left\{P\geqq 0: \|A-P\|=\delta\right\}$$

and

$$\mathscr{P}_n(A)=\left\{P\geqq 0: |||A-P|||=\delta\right\}.$$

Then both $\mathscr{P}(A)$ *and* $\mathscr{P}_n(A)$ *are convex sets and* $\mathscr{P}(A)\subseteqq\mathscr{P}_n(A)$. *The operators in* $\mathscr{P}(A)$ *and* $\mathscr{P}_n(A)$ *are called positive approximants and positive near-approximants respectively.*