On balanced projectives and injectives over linearly compact rings

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Dedicated Professor Kiiti Morita on his 60th birthday

(Received September 19, 1975)

Introduction

Let ${}_RM$ be a left R-module over a ring R[1] and C be the biendomorphism ring of ${}_RM$. Then there exists a canonical ring homomorphism δ of R into C which is defined by $\delta(r)(m)=rm$, $r\in R$, $m\in M$. ${}_RM$ is called balanced if δ is an epimorphism. It is shown that Morita-Suzuki's criterion[2] for δ to be an isomorphism is easily generalized for modules from the view point of reflexivity. Thus we have the following

THEOREM (THEOREM 1). *Let R, S be two rings. Let ${}_RX$ be a left R-module and ${}_RZ_S$ be a two-sided R-S-module. Then the following statements are equivalent*:

(1) *${}_RX$ is Z-reflexive.*

(2) (i) *The Z-dual of ${}_RX$ is Z-reflexive*

(ii) *There exists an exact sequence of left R-modules*

$$0\to X\to \Pi Z\to \Pi Z,$$

where ΠZ's denote the direct products of copies of Z, though the index sets are generally different[3].

Let ${}_RP$ be a finitely generated projective module and ${}_RQ$ be an injective module with essential socle such that each simple homomorphic image of ${}_RP$ is isomorphic to a submodule of ${}_RQ$ and each simple submodule of ${}_RQ$ is a homomorphic image of ${}_RP$. Let S and T be the endomorphism ring of ${}_RP$ and ${}_RQ$, respectively. Then the left S-module ${}_S\mathrm{Hom}_R(P, Q)$ is an injective cogenerator with the endomorphism ring T, and the $\mathrm{Hom}_R(P, Q)$-dual of ${}_SP^*=\mathrm{Hom}_R(P, R)$ is isomorphic to Q_T[4]. It is shown that

1) In what follows we assume that all rings have an identity element and all modules are unital.

2) Cf. [5].

3) That is, Z-dominant dimension of $X\geqq 2$ in the terminology of [5].

4) See Lemma 3 and Theorem 1, [8]. There one can easily replace cofinitely generated injectiveness for ${}_RQ$ by injectiveness with essential socle, as T. Kato pointed out to the author.