Note on simple ring extensions

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Throughout, A/B will represent a ring extension of Artinian simple rings (with 1), V the centralizer of B in A, and $A^*=\mathrm{Hom}({}_BA, {}_BB)$. In this note, we shall prove the following:

THEOREM. *Assume that* $[A:B]_l<\infty$ *and* A *is* BV-A-*irreducible and* A-BV-*irreducible. If* $\mathrm{Hom}({}_AA^*{}_B, {}_AA_B)\neq 0$ *then* A/B *is a Frobenius extension.*

PROOF. First, we claim that

$$(1)\qquad \mathrm{Hom}({}_AA_A, {}_AA\otimes_BA_A)\cong \mathrm{Hom}({}_AA^*{}_B, {}_AA_B)\cong \mathrm{Hom}({}_A(\mathrm{End}_BA)_A, {}_AA_A).$$

In fact,

$${}_AA\otimes_BA_A\cong {}_A\mathrm{Hom}(B_B, A_B)\otimes_BA_A\cong {}_A\mathrm{Hom}(A^*{}_B, A_B)_A$$

and

$${}_AA\otimes_BA_A\cong {}_A\mathrm{Hom}(A_A, A_A)\otimes_BA_A\cong {}_A\mathrm{Hom}((\mathrm{End}_BA)_A, A_A)_A.$$

Hence, $\mathrm{Hom}({}_AA_A, {}_AA\otimes_BA_A)\cong\{u\in A\otimes_BA\,|\,au=ua \text{ for all } a\in A\}\cong\mathrm{Hom}({}_AA^*{}_B, {}_AA_B)$ resp. $\mathrm{Hom}({}_A(\mathrm{End}_B\,A)_A, {}_AA_A)$.

To be easily seen, ${}_AA_B$ and ${}_BA_A$ are homogeneously completely reducible and their lengths coincide with the capacity of the simple ring V[1]. Then, from $\mathrm{Hom}({}_AA^*{}_B, {}_AA_B)\neq 0$ one will easily see that there exists an epimorphism h: ${}_AA^*{}_B\to {}_AA_B$. It follows then $[A:B]_l=[A^*:B]_r\geqq[A:B]_r$. In particular, there holds the symmetric statment of (1) and $\mathrm{Hom}({}_B(\mathrm{Hom}(A_B, B_B))_A, {}_BA_A)\neq 0$, which enables us to obtain $[A:B]_r\geqq[A:B]_l$, namely, $[A:B]_r=[A:B]_l=[A^*:B]_r$. Therefore, h is an isomorphism and A/B is a Frobenius extension.

COROLLARY 1. *Assume that* $[A:B]_l<\infty$ *and* A *is* BV-A-*irreducible and* A-BV-*irreducible. If* A/B *is a separable extension then it is a Frobenius extension.*

PROOF. There exists an $e=\sum x_i\otimes y_i\in A\otimes_BA$ such that $\sum x_iy_i=1$ and $ae=ea$ for all $a\in A$ (cf. for instance [1, p. 366]). Therefore, $\mathrm{Hom}({}_AA^*{}_B, {}_AA_B)\cong\mathrm{Hom}({}_AA_A, {}_AA\otimes_BA)\neq 0$ by (1) and A/B is a Frobenius extension.

Finally, if $\mathrm{End}\,{}_BA$ possesses a right free A-basis $\{\alpha_1, \cdots, \alpha_n\}$ such that

1) See [3, Proposition 5.4].