Complex parallelizable manifolds

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(Received April 30, 1975)

§1. Introduction. H. C. Wang studied a complex parallelizable manifold, which can be considered as the natural counterpart, in the theory of complex manifolds, of an ordinary completely parallelizable manifold, and obtained the following result [2][1]:

A compact complex parallelizable manifold can be regarded, up to a holomorphic homeomorphism, as the the compact coset space of a complex Lie group by a discrete isotropic subgroup and becomes Kählerian if and only if it is a complex torus.

The purpose of the present paper is to study the geometric properties of complex parallelizable manifolds without assumption of compactness. The method of investigation is the same as in the theory of extended Lie systems [5], [6], that is to express all the geometric quantities in terms of the scalars of structure C^a_{bc} and their conjugates (see §2). We introduce, in §3, a Riemannian metric g on a complex parallelizable manifold M and show that a condition for M to be Kählerian is that $C^a_{bc}=0$. Consequently the above result of Wang can be stated as follows:

A connected complete complex parallelizable manifold M can be regarded as the coset space of a complex Lie group by a discrete subgroup if and only if all the scalars C^a_{bc} are constant. And M is Kählerian with respect to the g if and only if $C^a_{bc}=0$.

The real version of the above first statement, of course, holds good [3]. In §4, we deal with the special vector fields i.e. Killing, conformal Killing, divergence-free and harmonic vector fields. As a consequence we have that some special parallelizations give rise to $C^a_{bc}=0$. Finally we prove in §5 that any holomorphic sectional curvature is non-positive at every point of M and so is the scalar curvature.

§2. Complex parallelization. Let M be an n-dimensional complex manifold with a complex structure J. Then M is called a complex parallelizable manifold if there exist, on M, n holomorphic vector fields linearly independent everywhere. We denote the vector fields and their complex conjugates by $Z_1, Z_2, \cdots, Z_n$ and $\bar{Z}_1, \bar{Z}_2, \cdots, \bar{Z}_n$ respectively. We set

1.) Numbers in brackets refer to the references at the end of the paper.