Hokkaido Mathematical Journal Vol. 6 (1977) p. 66-68

On a theorem of S. Chowla

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(Received May 4, 1976)

Let p be an odd prime. Then S. Chowla [3] proved the following theorem.

THEOREM. *The* $\frac{p-1}{2}$ *real numbers* $\cot(2\pi a/p)$, $a=1,2,\cdots,\frac{p-1}{2}$ *are linearly independent over the field* $\boldsymbol{Q}$ *of rational numbers.*

Other proofs were given by Hasse [4], Iwasawa [5] and by Ayoub [1], [2].

In this note, we shall show the following theorem, which is a generalization of the above theorem, by means of improving the method of Chowla's proof.

THEOREM. *Let* n *be an integer with* $n>2$ *and let* T *be a set of representatives* mod n *such that the union* $\{T, -T\}$ *is a complete set of residues prime to* n. *Then the* $\phi(n)/2$ *real numbers* $\cot(\pi a/n)$, $a\in T$ *are linearly independent over* $\boldsymbol{Q}$, *where* $\phi(n)$ *is the Euler totient function.*

Proof. Let D be the set of all Dirichlet characters to the modulus n. For a map

$$F:\ (\boldsymbol{Z}/n\boldsymbol{Z})^{\times}\longrightarrow \boldsymbol{C}$$

from the multiplicative group $(\boldsymbol{Z}/n\boldsymbol{Z})^{\times}$ of the residue class ring $\boldsymbol{Z}/n\boldsymbol{Z}$ to the complex field $\boldsymbol{C}$, we define the Fourier transform by

$$\hat{F}(\chi)=\frac{1}{\phi(n)}\sum_{\substack{a(\bmod n)\\(a,n)=1}}F(a)\bar{\chi}(a)\qquad(\chi\in D).$$

Then the inversion formula

$$F(a)=\sum_{\chi\in D}\hat{F}(\chi)\chi(a)\qquad(a\in Z,\ (a,n)=1)$$

holds.

We define

$$H(a)=-\frac{1}{n}\sum_{x=1}^{n-1}e^{-2\pi iax/n}\log(1-e^{2\pi ix/n})\qquad(a\in Z).$$

The formulas (6) and (16) in Lehmer [7] yield

$$\hat{H}(\chi_0)=\frac{1}{n}\sum_{p|n}\frac{\log p}{p-1},$$