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A Lindelöf type theorem on a Riemann surface

By Yukio NAGASAKA

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1. Introduction and definitions

Let $f(z)$ be a bounded analytic function in $|z|<1$. If $f(z)$ has an asymptotic value α along some path in $|z|<1$ terminating at $e^{i\theta_0}$, then $f(z)$ has necessarily the angular limit α at $e^{i\theta_0}$ (Lindelöf's theorem). In this paper we study Lindelöf type theorem for an analytic mapping from a hyperbolic Riemann surface into another Riemann surface.

Let R be a hyperbolic Riemann surface. For a positive superharmonic function s on R and a closed set F in R, we denote by $s_F^R=s_F$ the lower envelope of the family of all positive superharmonic functions s' on R with $s'(z)\geq s(z)$ quasi-everywhere on F. Then s_F is superharmonic on R. Let γ: $z=z(t)$, $0\leq t<1$, be an arc in R such that γ tends to the ideal boundary of R as $t\rightarrow 1$. This means that for every compact set K in R there exists $t_0=t_0(K)$, $0<t_0<1$, with $\{z(t)|t_0\leq t<1\}\subset R-K$. Let $\{R_n\}_{n=1}^{\infty}$ be an exhaustion of R. For $0<\delta<1$, we set

$$\Omega_n(\gamma\,;\,\delta)=\left\{z\in R\,\middle|\,1_{\gamma\cap(R-R_n)}(z)>\delta\right\},$$

$$\Omega^*(\gamma\,;\,\delta)=\left\{z\in R\,\middle|\,1_\gamma(z)>\delta\right\},\ \Omega_n^*(\gamma\,;\,\delta)=\Omega^*(\gamma\,;\,\delta)\cap(R-\bar{R}_n).$$

Then $\bigcap_{n=1}^{\infty}\Omega_n^*(\gamma\,;\,\delta)=\phi$. If $\lim_{n\rightarrow\infty}1_{\gamma\cap(R-R_n)}(z)\not\equiv 0$, $\bigcap_{n=1}^{\infty}\Omega_n(\gamma\,;\,\delta)\neq\phi$. Let $\phi:R\rightarrow X$ be an arbitrary mapping from R into a compact metric space X. We define the following cluster sets:

$$\phi(\gamma\,;\,\delta)=\bigcap_{n=1}^{\infty}\overline{\phi(\Omega_n(\gamma\,;\,\delta))},\ \phi_\Delta(\gamma)=\bigcup_{0<\delta<1}\phi(\gamma\,;\,\delta),$$

$$\phi^*(\gamma\,;\,\delta)=\bigcap_{n=1}^{\infty}\overline{\phi(\Omega_n^*(\gamma\,;\,\delta))},\ \phi_\Delta^*(\gamma)=\bigcup_{0<\delta<1}\phi^*(\gamma\,;\,\delta).$$

In §2, we show a relation between $\phi_\Delta(\gamma)$ and $\phi_\Delta^*(\gamma)$. If R is an open unit disk $\{|z|<1\}$ and $\gamma_\theta: z=z_\theta(t)=te^{i\theta}$, $0\leq t<1$, then $\phi^*(\gamma_\theta\,;\,\delta)$ coincides with an angular cluster set at $e^{i\theta}$ and $\phi_\Delta^*(\gamma_\theta)$ coincides with the outer angular cluster set at $e^{i\theta}$. Let $\phi(z)$ be an analytic mapping from R into another hyperbolic Riemann surface R', let R'^* be a metrizable compactification of R' and let $\lim_{t\rightarrow 1}\phi(z(t))=b\in R'^*$. We set