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A commutativity theorem for left s-unital rings

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Throughout R will represent a ring, and N the set of all nilpotents of R. Following [5], R is called a left s-unital ring if for each $x\in R$ there exists an element e such that $ex=x$. As was shown in [5, Theorem 1], if F is a finite subset of a left s-unital ring R then there exists $e\in R$ such that $ex=x$ for all $x\in F$.

In this note, we consider the following conditions:

1) For each $x\in R$ there exists a positive integer n such that $x-x^{n+1}\in N$.

1′) For each $x\in R$ there exist positive integers m and n such that $x^{m}-x^{m+n}\in N$.

1″) For each $x\in R$ there exist a positive integer n and an element x' in the subring $[x]$ generated by x such that $x^{n}=x^{n+1}x'$.

1‴) For each $x\in R$ there exists an element $x'\in[x]$ such that $x-x^{2}x'\in N$.

2) $x-y\in N$ and $y-z\in N$ imply that $x^{2}=z^{2}$ or $xy=yx$.

2_{*}) $x-y\in N$ implies that $x^{2}=y^{2}$ or both x and y are contained in the centralizer $V_{R}(N)$ of N in R.

2^{*}) $x-y\in N$ implies that $x^{2}=y^{2}$ or $xy=yx$.

In general, $2_{*})\Rightarrow 2)\Rightarrow 2^{*})$, and $1)\Rightarrow 1')\Rightarrow 1'')\Leftarrow 1''')$. Moreover, $1'')\Rightarrow 1''')$. In fact, for any $x'\in[x]$ we have $(x-x^{2}x')^{n}=(x^{n}-x^{n+1}x')-(x^{n}-x^{n+1}x')x''$ with some $x''\in[x]$.

Recently, in [1, Theorem 2], we proved that if a left s-unital ring R satisfies 1) and 2) then R is commutative. More recently, in [3, Theorem 2], D. L. Outcalt and A. Yaqub have proved that if a ring R with a left identity satisfies 1″) and 2_{*}) then R is commutative. It is the purpose of this note to present the following theorem which includes [1, Theorem 2] as well as [3, Theorem 2].

THEOREM. *Let R be a left s-unital ring satisfying* 2). *Then each of* 1), 1′), 1″), 1‴) *implies others and that R is commutative.*

The next easy lemma is included in [1, Lemma 1].

LEMMA 1. *Assume that R satisfies* 2^{*}).

(1) *$x^{2}\in V_{R}(N)$ for each $x\in R$, especially every idempotent is central.*