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The growth of entire and meromorphic functions

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1. Let $f(z)$ be an entire or meromorphic function. We shall denote by $\boldsymbol{C}$ the complex plane and by $\bar{\boldsymbol{C}}$, the extended complex plane. For $a\in\bar{\boldsymbol{C}}$, let $n(r, a)$ be the number of zeros of $f(z)-a$ in $|z|\leq r$, where for $a=\infty$, $n(r, \infty)$ stands for the number of poles of $f(z)$ in $|z|\leq r$. We shall assume, without loss of generality, that $f(z)$ has no zeros or poles at the origin. Let $T(r)=T(r,f)$ be the Nevanlinna characteristic function of $f(z)$. Let $n(0, a)=0$ and let

$$N(r, a,f)=N(r, a)=\int_{0}^{r}\frac{n(t, a)}{t}dt .$$

Let ρ be the order of $f(z)$. If

$$\limsup_{r\rightarrow\infty}\frac{\log^{+}n(r, a)}{\log r}=\rho_{1}(a,f)=\rho_{1}(a)<\rho ,$$

we call a an *e. v. B.* (exceptional value in the sense of Borel). If

$$1-\limsup_{r\rightarrow\infty}\frac{N(r, a)}{T(r,f)}=\delta(a,f)=\delta(a)>0 ,$$

a is called *e. v. N.* (exceptional value in the sense of Nevanlinna). Also, if

$$\tau=\limsup_{r\rightarrow\infty}\frac{T(r,f)}{r^{\rho}}(0<\rho<\infty) ,$$

then $f(z)$ is said to be of maximal, mean or minimal type according as $\tau=\infty$, $0<\tau<\infty$ or $\tau=0$. If $f(z)$ is an entire function, we denote as usual

$$M(r)=M(r,f)=\max_{|z|=r}|f(z)| .$$

2. We prove

THEOREM 1. *Let $f(z)$ be an entire function of order ρ, $(0\leq\rho<\infty)$. Then for every $\varepsilon>0$, as $r\rightarrow\infty$*

$$M\left(r+\frac{1}{r^{\rho-1+\varepsilon}}\right)\sim M(r) . \qquad (1)$$

PROOF: Since $\log M(r)$ is a convex function of $\log r$,