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The factorization in the commutant of a unitary operator

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1. Introduction.

In this paper we generalize the results concerning the factorization of positive (i. e. positive semidefintie) operator valued functions on the unit circle to the abstract context. Let $\mathscr{L}$ be a complex Hilbert space, U a unitary operator on $\mathscr{L}$ and $\mathscr{H}$ a closed subspace of $\mathscr{L}$ which is invariant under U. Let $\{U\}'$ denote the commutant of U and $\mathscr{A}$ the algebra consisting of all bounded operators A in $\{U\}'$ such that $A\mathscr{H}\subseteq\mathscr{H}$. We ask the following question; which positive operator T in $\{U\}'$ is factorable in the sense that $T=A^{*}A$ for some A in $\mathscr{A}$?

Let us recall a classical example. Let $\mathscr{E}$ be a separable Hiblert space, $L_{\mathscr{E}}^{2}$ the Hilbert space of all Lebesgue measurable $\mathscr{E}$-valued functions on the unit circle having square-integrable norm, and U_{0} the bilateral shift on $L_{\mathscr{E}}^{2}$, i. e. $(U_{0}f)(e^{i\theta})=e^{i\theta}f(e^{i\theta})$. Also let $L_{\mathscr{B}(\mathscr{E})}^{\infty}$ denote the algebra of all Legesgue measurable, essentially bounded functions from the unit circle to the algebra $\mathscr{B}(\mathscr{E})$ of bounded operators on $\mathscr{E}$, and M_{F} the multiplication operator on $L_{\mathscr{E}}^{2}$ by F in $L_{\mathscr{B}(\mathscr{E})}^{\infty}$, i. e. $(M_{F}f)(e^{i\theta})=F(e^{i\theta})f(e^{i\theta})$. It is known that the map $F\rightarrow M_{F}$ is a *-isomorphism from the algebra $L_{\mathscr{B}(\mathscr{E})}^{\infty}$ with involution $F^{*}(e^{i\theta})=(F(e^{i\theta}))^{*}$ onto the commutant $\{U_{0}\}'$ of U_{0}. (See, for example, [6, P48 and P50]). Let $H_{\mathscr{E}}^{2}$ and $H_{\mathscr{B}(\mathscr{E})}^{\infty}$ be the Hardy subspaces of $L_{\mathscr{E}}^{2}$ and $L_{\mathscr{B}(\mathscr{E})}^{\infty}$ respectively. It is easy to see that A lies in $H_{\mathscr{B}(\mathscr{E})}^{\infty}$ if and only if M_{A} maps $H_{\mathscr{E}}^{2}$ into itself. Thus the above question is essentially the factorization problem for positive operator valued functions if $\mathscr{L}=L_{\mathscr{E}}^{2}$, $\mathscr{H}=H_{\mathscr{E}}^{2}$ and $U=U_{0}$.

The above question was considered by Page and Gellar, in [5] and [2]. In [5], Page studies the invertibility of an operator $PA|\mathscr{H}$, where A lies in $\{U\}'$ and P is the orthogonal projection of $\mathscr{L}$ onto $\mathscr{H}$, and showed that every invertible positive operator in $\{U\}'$ is factorable. Subsequently Gellar and Page [2] generalized this result, but only in an unsatisfactory way.

In the present paper we first prove a theorem which gives necessary and sufficient conditions for factorability. This contains the theorem of