Hokkaido Mathematical Journal Vol. 9 (1980) p. 18-35

Fixed points for mappings majorized by real functionals

By Mau-Hsiang SHIH[1]

(Received April 30, 1979)

Abstract.

In this paper the author extends the Caristi-Kirk fixed point theorem to the multi-valued case, as well as give a generalization to the so-called "strictly p-nonexpansive mappings." Specialization yields a stronger form of Edelstein's fixed point theorem. A characterization of asymptotic regularity in terms of fixed point features of cluster points is made. Also given is a nonlinear generalization of Stein's theorem for spectral radius less than one. Furthermore, it is shown that the Browder-Göhde-Kirk fixed point theorem cannot be extended in a "natural way."

1. Introduction.

In recent papers [11], [12], [22] (see also [4], [5], [9], [15], [16], [29], [32]), Caristi and Kirk have shown the following interesting technical sharpenings of the Banach contraction mapping principle: Let (M, d) be a complete metric space and $f: M \rightarrow M$ an arbitrary mapping. Suppose that there exists a lower semi-continuous real functional $p: M \rightarrow [0, \infty)$ such that for each x in M,

$$d\big(x, f(x)\big) \leqslant p(x) - p\big(f(x)\big).$$

Then f has a fixed point. As pointed out by Kirk and Caristi [22], the strength of this result lies in fact that it typically applies to mappings f which need not be continuous. Caristi-Kirk's theorem includes the Banach contraction theorem as a very special case (by takin $p(x) = (1-k)^{-1} d(x, f(x))$ if $k < 1$ is a Lipschitz constant for f). Because this theorem can be applied to investigate the theory of normal solvability ([7], [8], [22], [23]) and of inward mappings ([11], [12]), Browder [9] pointed out this theorem may well become an important tool in the future development of nonlinear functional analysis. It is the purpose of the present paper to extend this result to

1 The author is deeply indebted to Professor Ky Fan for his constant inspiration, great encouragement and helpful suggestions.