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# On the Hall-Higman and Shult theorems (II)

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We use the same notation as in the preceding paper [5], Theorem 1 (a), (b), (c). In that paper, to prove Hall-Higman's and Shult's theorems, we noticed that the proof is reduced to the case where $V$ is a $kQ$-irreducible ([5], Hypothesis 1 (3)) But the proof of this fact is not trivial. The most well-known method to show this is to use the fact that the projective representations of cyclic groups are of degree one ([4], p. 704). See also [2], p. 363. We will give an easy proof of the following well-known theorem.

THEOREM A. *Let $G$ be a finite group, $Q$ a normal subgroup of $G$, $k$ an algebraic closed field, and $V$ an irreducible $kG$-module of finite dimensional such that $V_Q$ is a direct sum of isomorphic irreducible $kQ$-modules. Assume that $G/Q$ is cyclic. Then $V_Q$ is $kQ$-irreducible.*

REMARK. The conclusion holds even if $G/Q$ is a $p$-group and $char(k)=p$. The proof is similar as cyclic case.

LEMMA. *Every $k$-algebra automorphism of $M(n, k)$, the $k$-algebra of all $n\times n$ matrices over a field $k$, is inner.*

This lemma is a particular case of a well-known theorem of Skolem-Nother. This theorem and its proof are found in [3], Cor. of Th. 4. 3. 1 and [1], § 10. 1. In the present case, the proof of this lemma is easy. For example, use the fact that if $U$ is a vector space over $k$ of dimensional $n$ and $f$ is a $k$-algebra automorphism of $E=End_k(U)\cong M(n, k)$, then $U\cong Uf$ as $E$-modules.

We can now prove the theorem. Assume that $V_Q$ is the direct sum of $n$ isomorphic irreducible $kQ$-modules $W_1, \cdots, W_n$. Set $E=End_{kQ}(V_Q)$. Then $E\cong M(n, k)$ as $k$-algebras, because $\mathrm{Hom}_{kQ}(W_i, W_j)\cong k$ for any $i$, $j$ by Schur's lemma. Remember that $k$ is algebraic closed. Each element $x$ of $G$ induces a $k$-algebra automorphism of $E$ by $(v)f^x=(vx^{-1})fx$ for $v\in V$, $f\in E$, and so $E$ is a $kG$-module. Let $Z$ be the center of $E$, so that $Z$ consists of all scalar transformations, and so $Z\cong k$. Let $x$ be an element of $G$ which, together with $Q$, generates $G$. Since $Q$ acts trivially on the $kG$-module $E$ and $End_{kG}(V)=Z$ by Schur's lemma, we have that

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