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Some studies on group algebras

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In this paper we study a ring theoretical approach to the theory of modular representations of finite groups which is studied by several authors ([5], [6], [8], [9], [10], e. t. c.). Most of results in this note is not new but is proved by a character-free method.

Let F be a fixed algebraically closed field of characteristic p, a rational prime. If G is a finite group, we let FG denote the group algebra of G over F. If X is a subset of G, we let $\hat{X}$ be the sum of elements of X in FG. Other notations are standard and we refer to [2] and [5].

In section 1 we shall give a proof of the result of Brauer which appears in [1] without proof. In section 2, using results in section 1 we investigate the center of a group algebra and an alternating proof of the result of Osima [7] is given.

1. In this section we give a characterization of elements of the radical of a group algebra which appears in [1] without proof. Related results also appear in [12]. Let G be a finite group of order $p^{a}k$, $(p, k)=1$. Choose an integer b so that $p^{b}\equiv 1 \pmod{k}$ and $b\geqq a$. Let U be the F-subspace of FG generated by all commutators in FG. Then $U=\left\{\sum_{g\in G}a_{g}g\,;\ \sum_{g\in C}a_{g}=0 \text{ for every conjugacy class } C \text{ of } G\right\}$ and it holds that $(\alpha+\beta)^{p}\equiv\alpha^{p}+\beta^{p} \pmod{U}$ for α and β in FG. For these results see [2].

LEMMA (1. A). *Let $e=\sum_{g\in G}a_{g}g$ be an idempotent of FG. then we have $\sum_{g\in C}a_{g}=0$ for every p-singular conjugacy class C of G.*

PROOF. As $e^{p^{b}}=e$, we have $\sum_{g\in G}a_{g}g\equiv\sum_{g\in G}a_{g}^{p^{b}}g^{p^{b}} \pmod{U}$. Since coefficients of p-singular elements in the right-hand side of the above equation are all 0, we have the lemma.

LEMMA (1. B). *Let $e=\sum_{g\in G}a_{g}g$ be a primitive idempotent of FG. Then there exists a p'conjugacy class C of G such that $\sum_{g\in C}a_{g}\neq 0$.*

PROOF. Since e is primitive, $e\notin U$. Thus the lemma follows from (1. B).

Using the above lemmas, we can prove the assertion of Brauer stated in the begining of this section. Let $S_{1}, S_{2}, \cdots$ be p'-sections of G. If X and