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Some remarks on p-blocks of finite groups

By Tetsuro OKUYAMA

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In this paper we are concerned with modular representations of finite groups. Let G be a finite group and p a fixed rational prime. Let K be a complete p-adic field of characteristic 0 and R the ring of p-local integers in K with the principal maximal ideal (π) and the residue class field $R=R/(\pi)$ of characteristic p. We assume throughout the paper that fields K and R are both splitting fields for all subgroups of the given group G. We mention here [2] and [3] as general references for the modular representation theory of finite groups.

1. In this section we shall give some necessary and sufficient condition for G to be p-nilpotent. If B is a p-block of G, then let $Irr(B)$ denote the set of irreducible K-characters of G in B. For a class function θ of G we put $\theta_{B}=\sum_{\chi\in Irr(B)}(\theta, \chi)\chi$. Let $B_{0}(G)$ denote the principal p-block of G.

We prove the following.

THEOREM 1. *Let H be a subgroup of G which contains a Sylow p-subgroup P of G. If $1_{H}{}^{G}{}_{B_{0}(G)}(x)=1$ for any p-element $x\neq 1$ in G, then H controls the fusion of elements of P.*

To prove the theorem we use the following elementary lemma which follows from Brauer's Second Main Theorem.

LEMMA. *Let θ be a class function of G, x a p-element of G and B a p-block of G. Then $\theta_{B}(x)=\sum\theta_{|C_{G}(x)b}(x)$ where b ranges over the set of p-blocks of $C_{G}(x)$ with $b^{G}=B$.*

PROOF of THEOREM 1. Let $x\neq 1$ be an element of P, $C=C_{G}(x)$, $B=B_{0}(G)$ and $b=B_{0}(C)$. By Mackey decomposition we have $1_{H}{}^{G}{}_{|C}=\sum(1_{H^{y}\cap C})^{C}$ where y ranges over a complete set of representatives of (H, C)-double cosets in G. Thus the above lemma and the result of Brauer (Theorem 65. 4 [2]) show that $1_{H}{}^{G}{}_{B}(x)=\sum(1_{H^{y}\cap C})^{C}{}_{b}(x)$. If $x\in H^{y}\cap C$, then $(1_{H^{y}\cap C})^{C}{}_{b}(x)=(1_{H^{y}\cap C})^{C}{}_{b}(1)$. and if $x\notin H^{y}\cap C$, then $(1_{H^{y}\cap C})^{C}{}_{b}(x)=0$ by (6. 3) IV in [3]. As $1_{H}{}^{G}{}_{B}(x)=1$ by our assumption, $x\in H^{y}\cap C$ if and only if $y\in HC$. Therefore if $x^{y}\in H$ for some element y, then there exists an element h in H such that $x^{y}=x^{h}$ and therefore the theorem is proved.

As an easy corollary of Theorem 1 we have the following.