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Note on Hadamard matrices of Pless type

To Goro Azumaya on his sixtieth birthday

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Let H be an Hadamard matrix of order n. Namely H is a ± 1 matrix of degree n such that $HH^t = nI$, where t denotes the transposition and I is the identity matrix of degree n. We assume that $n>1$. It is well known that $n=2$ or n is divisible by 4.

Let $P=\{1, \cdots, n, 1^*, \cdots, n^*\}$ be the set of $2n$ points, where we assume that $(i^*)^*=i$ for $1 \leqq i \leqq n$. Then with each row vector $\boldsymbol{a}$ of H we associate the block $\boldsymbol{a}$, and n-subset of P, as follows. $\boldsymbol{a}$ contains j or j^* according as the j-th entry of $\boldsymbol{a}$ is 1 or -1. The complement $\boldsymbol{a}^*=P-\boldsymbol{a}$ of $\boldsymbol{a}$ is also called a block. Let B be the set of $2n$ blocks. Then we call $M(H)=(P, B)$ the matrix design of H. $M(H)$ is a 1-design, namely each point belongs to exactly n blocks. Moreover it is almost a symmetric 2-design. Namely by the orthogonality of column vectors of H each 2-subset of P not of the form $\{i, i^*\}$ is contained in exactly $\frac{1}{2}n$ blocks, while $\{i, i^*\}$ is contained in no blocks.

Let $G(H)$ be the set of all permutations $\boldsymbol{s}$ on P such that (i) $\boldsymbol{s}(B)=B$ and that (ii) if $\boldsymbol{s}(a)=b$ then $\boldsymbol{s}(a^*)=b^*$. Then $G(H)$ forms a subgroup of the symmetric group on P, namely the automorphism group of H. Let $\boldsymbol{z}=\prod_{i=1}^{n}(i, i^*)$. Then $\boldsymbol{z}$ belongs to the center of $G(H)$ and it interchanges $\boldsymbol{a}$ with $\boldsymbol{a}^*$ for every $\boldsymbol{a}$. We call $\boldsymbol{z}$ the $*$-element of $G(H)$.

Now the purpose of this note is the following: (i) to show that an Hadamard matrix of order $2(q+1)$, where q is a prime power with $q \equiv 3 \pmod 4$, constructed by V. Pless in [6], $H_3(q)$ in her notation, which we call an Hadamard matrix of Pless type, is inequivalent to the Hadamard matrix of order $2(q+1)$ of Paley type, provided that $q>3$. It is well known that there exists exactly one equivalent class of Hadamard matrices of order 8; (ii) to determine the automorphism groups of two types of Hadamard matrices of degree $2(q+1)$ mentioned above.