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On a special type of Galois extensions

Dedicated to Professor G. Azumaya on his 60th birthday

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1. In order to generalize the notion of Azumaya algebra, we researched on a special type of separable extension, called H-separable extension, and have found that many properties which hold in Azumaya algebras hold also in H-separable extensions (see for example [5], [2], [7] and [8]). In this paper we shall study relations between Galois extensions and H-separable extensions, and shall obtain some necessary and suficient conditions for Galois extensions to be H-separable extensions. By the definition of Galois extension and by Cor. 1.1 [5], we can easily see that in the case of algebras over a commutative ring R, H-separable Galois extensions of R is same as central Galois extension of R. Throughout this paper Λ shall always be a ring with 1, Γ a subring of Λ which contains same 1, C the center of Λ and $\Delta=V_{\Lambda}(\Gamma)=\Lambda^{\Gamma}$, and $M^{\Lambda}=\{m\in M|xm=mx$ for all $x\in\Lambda\}$ for any $\Lambda-\Lambda$-module M.

2. First, we shall recall definitions.

DEFINITION 1. Λ is called an H-separable extension of Γ when Λ and Γ satisfy one of the following equivalent conditions;

(a) $\Lambda\otimes_{\Gamma}\Lambda$ is isomorphic to a direct summand of $\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda$ (finite direct sum) as $\Lambda-\Lambda$-module.

(b) Δ is C-finitely generated projective, and the following map η is an isomorphism

$$\eta:\ \Lambda\otimes_{\Gamma}\Lambda\rightarrow\mathrm{Hom}\,({}_{C}\Delta,{}_{C}\Lambda)\quad \eta(x\otimes y)(d)=xyd\ \ (x,y\in\Lambda,d\in\Delta)$$

(c) For any $\Lambda-\Lambda$-module M, the following map g_{M} is an isomorphism

$$g_{M}:\ \Delta\otimes_{C}M^{\Lambda}\rightarrow M^{\Gamma}\quad g_{M}(d\otimes m)=dm\ \ (d\in\Delta,m\in M^{\Lambda})$$

(d) $1\otimes 1\in\Delta(\Lambda\otimes_{\Gamma}\Lambda)^{\Lambda}$

As for the proof of equivalence of (a)~(d), see Theorem 1.2 [5], Prop. 1 [6] or (1.3) [7]. Note that Azumaya algebra always satisfies these conditions.

Next, let $\mathfrak{G}$ be a finite group of automorphisms of Λ which fix all elements of Γ. We can make $\Sigma_{\sigma\in\mathfrak{G}}\Lambda U_{\sigma}$ a ring by $(xU_{\sigma})(yU_{\tau})=x\sigma(y)U_{\sigma\tau}$ $(\sigma,\tau\in\mathfrak{G})$, where $\{U_{\sigma}\}_{\sigma\in\mathfrak{G}}$ is a free Λ-basis. We denote this ring by $\Delta(\Lambda:\mathfrak{G})$. Then we can always have a ring homomorphism j of $\Delta(\Lambda:\mathfrak{G})$ to $\mathrm{Hom}\,(\Lambda_{\Gamma},\Lambda_{\Gamma})$ such that $j(xU_{\sigma})(y)=x\sigma(y)$ $(x,y\in\Lambda,\sigma\in\mathfrak{G})$.

DEFINITION 2. We say that Λ is a $\mathfrak{G}$-Galois extension of Γ when Λ