Hokkaido Mathematical Journal Vol. 10 (1981) p. 254-263

On the Jacobson radical of the center of an infinite group algebra

Dedicated to Professor Goro Azumaya on the occasion of his 60th birthday

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(Received September 22, 1979)

Throughout K will represent an algebraically closed field of characteristic $p>0$, and G a group. We denote by G', $Z(G)$ and P the commutator subgroup, the center and a Sylow p-subgroup of G respectively. For $x\in G$, C_x is the conjugacy class of G containing x. Given a finite subset S of G, we denote by $\hat{S}$ the element $\sum_{x\in S}x$ of the group algebra KG. If R is a ring (with identity), then $Z(R)$ and $J(R)$ denote the center and the (Jacobson) radical of R respectively, and $N(R)$ is the sum of all the nilpotent ideals of R.

In case G is a finite p-solvable group, R. J. Clarke [1] gave a necessary and sufficient condition for $J(Z(KG))$ to be an ideal of KG. Recently, S. Koshitani [2] proved that if G is finite and $J(Z(KG))$ is an ideal of KG then G is p-solvable. Hence, in case G is finite, the problem to find a necessary and sufficient condition for $J(Z(KG))$ to be an ideal of KG has been solved completely. In this paper, we consider this problem for infinite groups, and give an answer for poly-$\{p, p'\}$ groups.

At first we recall the following

THEOREM 1 (Passman [5, Lemma 4. 1. 11]). $J(KG)\cap Z(KG)=J(Z(KG))$.

Now, by making use of the same argument as in the proof of [1, Lemma 4], we shall prove the next

LEMMA 1. *Suppose that $J(Z(KG))$ is an ideal of KG. Then the following statements hold*:

(1) *If G' is an infinite group, then $J(Z(KG))=0$.*

(2) *If G' is a finite group with $p\nmid|G'|$, then $J(Z(KG))=\hat{G}'J(KG)$.*

(3) *If G' is a finite group with $p\mid|G'|$, then $J(Z(KG))=\hat{G}'KG$.*

PROOF. Since $J(Z(KG))$ is an ideal of KG, for x, $y\in G$ and $a\in J(Z(KG))$ we have

$$(x^{-1}y^{-1}xy)\,a=x^{-1}y^{-1}(ya)\,x=x^{-1}ax=a\,.$$

Hence $ga=a$ for all $g\in G'$. Therefore it is easily seen that if G' is infinite