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On the construction of p-adic L-functions

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Let $\boldsymbol{Q}$ be the rational number field, $\bar{\boldsymbol{Q}}$ the algebraic closure of $\boldsymbol{Q}$, $\boldsymbol{C}$ the complex number field, p a prime number, $\boldsymbol{Q}_p$ the p-adic rational number field, $\boldsymbol{Z}_p$ the integer ring of $\boldsymbol{Q}_p$, $\boldsymbol{C}_p$ the completion of the algebraic closure of $\boldsymbol{Q}_p$ and let $\mathfrak{m}$ be the maximal ideal of the integer ring of $\boldsymbol{C}_p$. We fix an imbedding of $\bar{\boldsymbol{Q}}$ into $\boldsymbol{C}$ and also fix an imbedding of $\bar{\boldsymbol{Q}}$ into $\boldsymbol{C}_p$. Let $L_i(z)=L_i(z_1, \cdots, z_r)=\sum_{1\leqq j\leqq r} a_{ij}z_j$ be linear forms of r variables, where i ranges from 1 to n, r and n are natural numbers. We suppose that the coefficients a_{ij} are algebraic numbers and satisfy the following conditions: a_{ij} are real positive when considered as complex numbers, and $a_{ij}\in\mathfrak{m}$ when considered as p-adic numbers. Let $L_j^*(t)=L_j^*(t_1, \cdots, t_n)=\sum_{1\leqq i\leqq n} a_{ij}t_i$ be linear forms with above coefficients a_{ij}, where j ranges from 1 to r.

In the following, let us agree that the suffix i ranges from 1 to n and the suffix j ranges from 1 to r. We also agree that an algebraic number may be considered both as a complex number and as a p-adic number by the above fixed imbeddings.

Let $\chi_j:(\boldsymbol{Z}/d_j\boldsymbol{Z})^\times\to\bar{\boldsymbol{Q}}^\times$ be Dirichlet characters defined modulo d_j, which may be not necessarily primitive (here $R^\times$ denotes the multiplicative group of invertible elements of a ring R and $\boldsymbol{Z}$ denotes the ring of rational integers). Let $\xi_j\in\bar{\boldsymbol{Q}}^\times$ be such that $\xi_j^{d_j}\not\equiv 1 \pmod{\mathfrak{m}}$ and $|\xi_j|\leqq 1$ where $|\xi_j|$ is the absolute value of ξ_j considered as a complex number. Let x_j be real algebraic number such that $0\leqq x_j<1$ and $L_i(x)\equiv 1 \pmod{\mathfrak{m}}$ for $i=1, \cdots, n$, where we have put $x=(x_1, \cdots, x_r)$.

Now we define a function $Z(s)=Z(s_1, \cdots, s_n)$ of n complex variables $s=(s_1, \cdots, s_n)$ by

$$Z(s)=\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{\chi_1(m_1)\cdots\chi_r(m_r)\,\xi_1^{m_1}\cdots\xi_r^{m_r}}{L_1(x+m)^{s_1}\cdots L_n(x+m)^{s_n}}$$

where $x+m=(x_1+m_1, \cdots, x_r+m_r)$.

It is easy to see that this series is absolutely convergent when the real parts of $s_1, \cdots, s_n$ are sufficiently large to give there a complex analytic function.

Next we define a mermorphic (i.e., meromorphic in each variable) function $G(t)=G(t_1, \cdots, t_n)$ of n complex variables $t=(t_1, \cdots, t_n)$ by