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Double integral theorem of Haar measures

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On a group G we consider only those uniformities $\boldsymbol{U}$ for which the right transformation group R_G is equi-continuous, i.e., for any $U\in\boldsymbol{U}$ there is $V\in\boldsymbol{U}$ such that $xVy\subset xyU$ for every $x, y\in G$. A set $A\subset G$ is said to be *totally bounded* for $\boldsymbol{U}$ if for any $U\in\boldsymbol{U}$ we can find a finite system $x_\nu\in G$ $(\nu=1, 2, \cdots, n)$ for which we have $A\subset\bigcup_{\nu=1}^{n}x_\nu U$. The linear lattice Φ of all uniformly continuous functions φ on G for which $\{x: \varphi(x)\neq 0\}$ are totally bounded for $\boldsymbol{U}$ is called the *trunk* of $\boldsymbol{U}$. A positive linear functional μ on Φ is called a *measure* on Φ and its value is denoted by $\int\varphi(x)\,\mu(dx)$ for $\varphi\in\Phi$.

For a transformation T on G, if both T and T^{-1} are uniformly continuous for $\boldsymbol{U}$, then for any $\varphi\in\Phi$, setting $\psi(x)=\varphi(xT)$ for $x\in G$, we obtain $\psi\in\Phi$. A measure μ on Φ is called a *Haar measure* of G for $\boldsymbol{U}$ if $\mu\neq 0$ and μ is invariant by R_G, i.e.,

$$\int\varphi(xy)\,\mu(dx)=\int\varphi(x)\,\mu(dx)\qquad\text{for}\quad\varphi\in\Phi\quad\text{and}\quad y\in G\,.$$

A uniformity $\boldsymbol{U}$ on G is said to be *locally totally bounded* if there is $U\in\boldsymbol{U}$ such that xU is totally bounded for every $x\in G$. According to the Theorem of Existence in [3], if $\boldsymbol{U}$ is locally totally bounded, then there is a Haar measure of G for $\boldsymbol{U}$. If every left transformation $L_x(X\in G)$ is uniformly continuous for $\boldsymbol{U}$ in addition, then we can apply the Theorem of Uniqueness in [3], and we have that the Haar measures are uniquely determined except for constant multiplication, i.e., for any two Haar measures μ and ρ there is a positive number α such that

$$\int\varphi(x)\,\mu(dx)=\alpha\int\varphi(x)\,\rho(dx)\qquad\text{for every}\quad\varphi\in\Phi\,.$$

For a topological group G we defined the proper uniformity on G in [6]. For the proper uniformity the right transformation group R_G is equi-continuous and every left transformation $L_x(x\in G)$ is uniformly continuous. Therefore for a locally compact topological group G there exists a Haar

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