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On the radical of the center of a group algebra

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1. Let kG denote the group algebra of G over a field k of characteristic $p>0$ and $Z=Z(kG)$, the center of kG. In this short note we shall prove the following;

THEOREM. *Let e be a block idempotent of kG with defect d. If $J(Z)$ denotes the Jacobson radical of Z, then the following hold;*

(1) $J(Z)^{p^{d}}e=0$.

(2) *If k is algebraically closed, then $J(Z)^{p^{d}-1}e\neq 0$ if and only if the block of kG corresponding to e is p-nilpotent with a cyclic defect group.*

As a corollary of the theorem we have the following which extends the result of Passman (Theorem, [8]);

COROLLARY. *Let $|G|=p^{a}m$ with $(p, m)=1$. Then $J(Z)^{p^{a}}=0$.*

2. Proof of the theorem.

To prove the theorem we may assume that k is algebraically closed (see Corollary 12. 12, [6] and (9. 10) Chapter III, [4]). We shall prove the statement (1) by induction on d. If $d=0$, then $J(Z)e=0$ and the result follows easily. Assume $d>0$ and we shall show;

(a). *Let x be a p-element of G of order p^{b}, $b>0$ which is contained in a defect group of e and σ the Brauer homomorphism from Z to $Z(kC_{G}(x))$. Then*

$$\sigma\left(J(Z)^{p^{d}-1}e\right)\subseteq\alpha kC_{G}(x)$$

where $\alpha=\sum_{i=0}^{p^{b}-1}x^{i}$.

PROOF of (a). Let f be a block idempotent of $kC_{G}(x)$ with $f\sigma(e)=f$. Then the defect of f is at most d (see § 9, Chapter III, [4]). Consider the homomorphism τ from $kC_{G}(x)$ onto $kC_{G}(x)/\langle x\rangle$ induced by the natural homomorphism of $C_{G}(x)$ to $C_{G}(x)/\langle x\rangle$. The kernel of τ is $(x-1)kC_{G}(x)$ and $\tau(f)$ is a block idempotent of $kC_{G}(x)/\langle x\rangle$ with defect at most $d-b$ (see § 4, Chapter V, [4]). Thus by induction it follows that $J(Z(kC_{G}(x)))^{p^{d-b}}f\subseteq (x-1)kC_{G}(x)$. Since $p^{d}-1\geq p^{d-b}(p^{b}-1)$, $J(Z(C_{G}(x)))^{p^{d}-1}f\subseteq((x-1)kC_{G}(x)^{p^{b}-1}$